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Theory of Massive Particles I. Algebraic Structure

G. A. J. Sparling

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THEORY OF MASSIVE PARTICLES
I. ALGEBRAIC STRUCTURE

By G. A. J. SPARLING
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260, *U.S.A.*

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CONTENTS

A theory of massive particles is developed from the single hypothesis of broken conformal invariance. The theory naturally incorporates fermions and bosons into a single framework. Applied to hadrons, it gives a structure conjectured to be that of the Regge trajectories, including the daughter particles. The theory predicts that the trajectories are exotic.

Introduction

The description of elementary particles within the framework of quantum mechanics and special relativity began with the discovery of the equation for a scalar field by de Broglie (1925), Schrödinger (1926*a*, *b*), Klein (1926), Fock (1926*a*, *b*) Kudar (1926), de Donder & van Dungen (1926) and Gordon (1926*a*, *b*). The lack of a suitable probability interpretation for the wave function led Dirac (1927) to construct a first-order relativistic wave equation describing a particle of spin one-half. His work showed that the idea of spin for a particle, first exploited by Heisenberg (1924), Uhlenbeck & Goudsmit (1925, 1926), Pauli (1927) and Darwin (1927), was fundamentally related to the Poincaré group and hence to the nature of space–time itself (Weyl 1931). Experimentally the equation led to a series of triumphs, the most dramatic, perhaps, being the confirmation of his suggestion that for every particle there should exist an anti-particle, whose basic properties could be completely predicted (Dirac 1930; Oppenheimer 1930; Anderson 1932). Also the simple form of the equation suggested how to couple systems described by it to electromagnetic fields, preserving relativistic invariance and leading eventually to the apparatus of modern quantum field theory.

Following Dirac's work, physicists have accepted that a minimal requirement for the external description of an elementary particle is that its wave function should belong to a unitary ray representation of the Poincaré group. Wigner effectively solved the problem of construction all such representations. More accurately, he reduced the problem to the task of finding representations of certain little groups, the isotropy groups of the Poincaré group when acting on various points in momentum space. For some of these groups the representation theory had not been finished. However, for a system of positive mass, the little group was just the rotation group, so Wigner (1939) provided a complete account.

In 1932, Heisenberg formulated the notion of internal symmetry. He showed that it was useful in analysing nuclear interactions to regard the proton and neutron as being essentially equivalent. Mathematically he regarded the neutron and proton as forming basis states for the fundamental representation of SU(2), the hypothesis being that nuclear forces were invariant under SU(2) or isospin rotations (Heisenberg 1932; Cassen & Condon 1936). This idea was generalised successfully by Gell-Mann & Ne'eman (1964) to an SU(3) invariance.

A curious feature of their approach was that the basic observed particles fitted not into the fundamental three-dimensional SU(3) representation but into the octet, the eight dimensional adjoint representation: 'the eightfold way'. Another striking feature of these basic particles, the pseudo-scalar mesons, the vector mesons, the baryons and the anti-baryons is that some are fermions and others bosons.

In the light of modern attempts to unify fermions and bosons within the framework of supersymmetry, one might wonder whether some such symmetry exists connecting the different octets. It is the basic contention of the following work that such a symmetry arises naturally in the context of broken conformal invariance.

Surprisingly this new symmetry is again (the Lie algebra of) SU(3). Quite unexpectedly it extends the relativistic spin SU(2) symmetry found by Dirac and Wigner in just the same way as the SU(3) of Gell-Mann & Ne'eman extends the isotopic SU(2) of Heisenberg. Analogous to the idea of hypercharge introduced by Gell-Mann, in this scheme, is the baryon number. Analogous to isospin is the ordinary relativistic spin.

The origin of the present theory lies in a new approach to answering the following question: what happens to a system transforming under the conformal group, when the conformal invariance is broken by fixing the mass?

As early as 1910 it was recognized by Cunningham (1910) and Bateman (1910) that Maxwell's equations were conformally invariant. As the study of quantum mechanics has become increasingly sophisticated, physicists have returned again and again to the conformal group as a mathematical tool in their analysis. The conformal group is particularly convenient mathematically since it is simple. However it has never achieved full respectability because its action on a quantum mechanical system changes its mass in a continuous manner (Mack & Salam 1969).

If one asks what parts of the conformal group or its Lie algebra commute with the mass, then the answer is simply the Poincaré group or its Lie algebra respectively. However a crucial assumption of quantum mechanics is that operators on a quantum system may be multiplied together in an associative way, not just commuted with each other. For a Lie algebra acting on a quantum system, this means that the action extends naturally to an action of the enveloping algebra of its Lie algebra. Mathematically, the enveloping algebra is the smallest associative algebra, extending the given Lie algebra, such that the Lie bracket becomes just the commutator of operators.

Remarkably, if one asks which sub-algebra of the enveloping algebra of the conformal algebra commutes with the mass, the answer is an algebra larger than the enveloping algebra of the Poincaré Lie algebra. More precisely a four-vector operator, called R^a below, must be added to the ten Poincaré generators to obtain the required algebra, here called the $\mathscr{R}$-algebra. Furthermore R^a turns out to commute not only with the mass, but also with all four translations. So R^a is an intrinsic operator associated to the system, like the intrinsic spin. R^a, the Poincaré algebra and the dilation generate the whole enveloping algebra of the conformal algebra.

The construction of the R^a operator reduces the analysis of representations of the conformal algebra to the discussion of representations of a 'little algebra' in complete analogy to the work of Wigner with the Poincaré group, except that the little algebra is not a Lie algebra. As in Wigner's work, the physically most interesting positive mass case only will be analysed exhaustively here, the massless case being already well known.

A key point is that when conformal invariance is broken, provided the R^a operator survives, essentially no useful information is lost. Although the dilations have been removed, even the information of the intrinsic dimension of the system is retained – in other words there is only one way of putting back the dilations.

In § 1 the definition and basic algebraic properties of the R^a operator are discussed. Then in § 2, the representation theory of the $\mathscr{R}$-algebra is obtained, using the 'little algebra' technique, for positive mass.

The representations are easily described. They consist of a collection of Poincaré representations all of the same (positive) mass and of spins $j\hbar$, satisfying $j_{\min} \leqslant j \leqslant j_{\max}$, $j - j_{\min}$ and $j_{\max} - j_{\min}$ non-negative integers, where $j_{\max}$ may be infinite. Each allowed spin occurs only once. If $j_{\max}$ is finite a further continuous real number $\alpha > j_{\max}$ is required to fix the representation completely. If $j_{\max}$ is infinite, two numbers are required. The vector part of R^a carries the spin j into $j \pm 1$ relating the different Poincaré representations.

Physically, it is conjectured that the representation of bounded spin corresponds to a hadron resonance of spin $j_{\max}$ and its daughters of lower spin, but roughly the same mass. This involves pretending that these resonances are stable – it is a zero width approximation. It will turn out that all the daughters have the same parity as their parent, contrary to one's naive expectation. The parameter α carries the information of the dimension of the system.

The structure of the R^a operator in these representations is too complicated for one to be completely satisfied. One can ask, can one factorize R^a, modulo terms in the spin and momentum, into the product of a spinor operator r^A and its Hermitian conjugate $r^{A'}$? Clearly the answer is no, within the context of a single $\mathscr{R}$-algebra representation, since r^A must carry half-integer spin, whereas the spins in an $\mathscr{R}$ representation are integrally spaced. Hence the r^A operator if it exists must be an intertwining operator connecting different $\mathscr{R}$ representations. In particular it must change the Casimir eigenvalues defining the $\mathscr{R}$ representation. To handle these changes neatly, the Casimir eigenvalues are expressed as symmetric polynomials in four variables k, l, m and p which sum to zero. Then it is found that an essentially unique r^A exists, with the required properties, where r^A takes k, l, m, p to $k+1, l+1, m+1, p-3$.

Surprisingly when the commutator properties of r^A with itself and its conjugate $r^{A'}$ are examined, one finds that r^A, $r^{A'}$, S^a and $\frac{1}{3}P$ generate $SU(3)$, where S^a is the relativistic spin operator introduced by Pauli and Lubánski (1942) and P is the operator with eigenvalue p. So the structure of the $\mathscr{R}$ representations may be subsumed in the well-known theory of SU(3) representations. It is hard to see how this result could have been anticipated.

To tie down the physics completely it is suggested that $\frac{1}{3}P$ is the baryon number and the SU(3) quantum numbers for the internal symmetry multiplet are identical to the SU(3) quantum numbers of the r^A, S^a, P representation.

In terms of the analogy with the SU(3) of the eightfold way, r^A is an operator carrying one unit of 'strangeness' (here baryon number) and 'isospin' one half (here spin one half). R^a is then an operator of zero 'strangeness' which mixes up the states of a given 'strangeness' in the SU(3) multiplet, carrying 'isospins' one and zero. The idea that R^a relates the resonance of highest spin to its daughters of lowest spin may now be generalized: the whole SU(3) multiplet consisting of states of a variety of spins and baryon numbers is to be considered as a unit. One may regard the state of highest spin as being the parent, all the others then being the daughters. However it is probably more democratic to regard all states as being on an equal footing: a nuclear family without a matriarch or patriarch!

The basic example of this structure is given by the octet. This contains eight states, two each of baryon numbers plus and minus one, four of baryon number zero. The two independent states of baryon number plus one are the spin-up and spin-down states of the *same spin one half particle*. Similarly for the two states of baryon number minus one. The four states of baryon number zero comprise the three spin states of a particle of spin one together with the single spin state of its 'daughter' particle of spin zero.

The hypothesis that the internal SU(3) representation is the same as the 'external' requires that each particle in the 'external' octet belongs to an internal SU(3) octet. For this fundamental octet, the obvious candidates are the (stable) baryon octet for the states of baryon number plus one, the vector and pseudoscalar meson octets for the states of baryon number zero and the anti-baryon octet for the states of baryon number minus one.

Note that unlike the situation for the internal SU(3), the external 'isopin' symmetry is completely unbroken, at least to the extent that space–time is locally isotropic. The external 'strangeness' is, on the other hand, totally broken.

In nature one is aware that resonances do not all exist, even approximately, at one given mass value. Indeed there is considerable evidence for the existence of sequences of states of increasing spin whose mass, squared, varies approximately linearly with the spin, the so-called Regge trajectories (see for example, Chew & Frautschi 1960; Collins 1977; Penrose *et al.* 1978). It is normally assumed that apart from their mass, spin and possibly parity, the states on a given trajectory have the same quantum numbers, in particular, the same SU(3) quantum numbers. However, in the present scheme, the non-negative integers λ and μ which characterize an SU(3) representation obey the inequalities

$$\lambda+\mu-|B-\tfrac{1}{3}(\mu-\lambda)| \geqslant 2J \geqslant |B-\tfrac{2}{3}(\lambda-\mu)|,$$

where B is interpreted as the baryon number and $J\hbar$ is the spin. In particular $\lambda+\mu \geqslant 2J$, so, on a trajectory, as J increases, so must $\lambda+\mu$.

Thus the main qualitative prediction of this theory is that Regge trajectories are exotic, the states of higher spin necessarily belonging to SU(3) multiplets which are not of the standard octet, decuplet or singlet types.

The question arises: is there an algebra capable of generating the entire structure of the Regge trajectory, rather than just the daughter structure? Since the trajectories are probably infinite, such an algebra would have to be non-compact. Surprisingly, a natural candidate for such an algebra is already at hand. One notices that the r^A operator is asymmetric in its treatment of k, l,

m and p. One asks if operators, s^A, t^A and λ^A, analogous to r^A exist that factorize R^a and that respectively lower one of k, l or m by three units and raise the remainder of k, l, m and p by one unit. One finds that such operators do exist, with one crucial proviso. Just as $R^a = r^A \bar{r}^{A'}$ modulo terms in the spin and momentum, one has $R^a = \lambda^A \bar{\lambda}^{A'}$, but one finds that $R^a = -s^A \bar{s}^{A'}$ and $R^a = -t^A \bar{t}^{A'}$, each modulo terms in the spin and momentum.

Each of λ^A, s^A, t^A generates an algebra analogous to the SU(3) found for the r^A operator: the λ^A algebra is again SU(3), but because of the minus signs above, the s^A and t^A algebras are SU(1, 2). Finally one asks what structure is generated by all the operators r^A, s^A, t^A and λ^A taken together. Amazingly one finds it is a Lie algebra of twenty-eight dimensions, the algebra called O(4, Q) or SO(6, 2). Its maximal compact subalgebra is SO(6) × SO(2) and is the algebra generated by r^A and λ^A together.

The entire structure then forms one hermitian representation of SO(6, 2) and is conjectured to be the collection of all particles along a given trajectory, including the 'daughter' or 'nuclear family' structure. The representation breaks up into irreducible SO(6) representations corresponding to symmetric trace-free k-index tensors in six dimensions, one for each non-negative integer k. Both particle and anti-particle appear in each such representation. It is conjectured that the overall mass formula has the square of the mass depending linearly on the parameter k, which turns out to be just $\lambda + \mu$.

For each fixed k, each pair (λ, μ) with sum k occurs once in the SO(6) representation. This means that the SO(6) representation, for $k > 0$, always contains SU(3) multiplets of non-zero triality. Whether these can exist or not is a moot point. Standard points of view suggest that such multiplets would have non-integral electric charge. It seems that one must reserve judgement on this issue, for the moment.

Even if they do not now exist, one might speculate that at an earlier stage in the universe's history, at a temperature high enough to possess symmetries which are now broken, such multiplets may well have existed. It is clearly very possible that at such times, the particle spectrum might have been radically different from the current spectrum.

Another interesting and seemingly natural possibility is that the 'true' internal symmetry group is not the SU(3) of Gell-Mann & Ne'eman (1964), but is SU(4) or U(4), to match the maximal compact subgroup of the spectrum generating algebra SO(6, 2). Thus the internal symmetry representation at level k might well be the space of symmetric trace-free k-index tensors in six dimensions, or equivalently the SU(4) representation corresponding to the $2 \times k$ rectangular Young tableau. Whether this SU(4) has anything to do with the new family of particles recently unearthed remains to be seen.

In §§ 3 and 4 the derivation of all the mathematical structure alluded to so far, will be given. It is to be noted that the discussion is entirely abstract and relies not at all on any model of the system under discussion. Only elementary quantum mechanics is used and rigorous mathematics. It is not even assumed that the system is to be described by quantum fields on space–time.

However, remarkably, there is at least one model, which possesses all the properties under discussion, known to this author. This is the twistor model of hadrons, which seeks to describe a hadron by three twistors and has been studied intensely over the past six years by Penrose (1975), Perjés (1975), Perjés & Sparling (1979), Hughston (1979), Hodges (1975), this author and others. In particular, the key property of the factorization of the R^a operator was discovered by the author some two years ago, in the twistor model, though stupidly it was not realized at that time that the r^A operator generated with the spin and the baryon number an SU(3) algebra. In the twistor model the r^A operator is a measure of the non-locality of the system. It vanishes identically, for a two-twistor system, which is thought to describe a lepton.

The twistor system has one further feature which the abstract picture currently does not: it automatically links the internal and external symmetry in such a way that the λ and μ for the external SU(3) are indeed identical to the internal SU(3) λ, μ quantum numbers. This is an elegant *ansatz* of the abstract picture, but a prediction of the twistor model.

The experimental test is basically whether or not the trajectories are exotic. If it turns out that they are, this will lend credence to both the abstract picture and the twistor model. It is not yet understood by this author whether or not there is an additional SU(4) internal symmetry hidden in the twistor scheme to match the full SO(6) sub-algebra of the spectrum generating algebra. One way to force such an SU(4) to exist is to go to a four-twistor scheme. But it is conceivable that it already exists in the three-twistor scheme if properly understood.

In §5 of part II of this work, to be published separately, some of the phenomenology is discussed, particularly indications already present in the data for the existence of exotic SU(3) multiplets. In §6 of part II, the description of the system by means of quantum fields is tackled. One begins to see that the existence of the r^A operator is connected with transformations that mix together space–time and spinors, giving a commutative version of supersymmetry.

It has long been held by this author and others, that the existence of such objects as the proton precludes the definition of points 'within' the proton. The proton is non-local in a very deep way, yet to be completely fathomed. Here one is beginning to see the emergence of a theory that will deal with these features directly.

1. The $\mathscr{R}$-algebra

From now onwards, since the introduction has been rather lengthy, the presentation will concentrate on the mathematics, keeping discussion to a minimum.

The standard abstract index convention of Penrose will be used throughout.

The infinitesimal generators of the conformal group of space–time form a fifteen-dimensional algebra, $\mathscr{C}$, spanned by the translations P_a, the Lorentz transformations $M_b{}^c = -M^c{}_b$, the dilation D and the special conformal transformations Q^a. P_a, $M_b{}^c$ and D together span the eleven-dimensional sub-algebra $\mathscr{D}$. P_a and $M_b{}^c$ span the ten-dimensional Poincaré subalgebra $\mathscr{P}$ of $\mathscr{C}$ and $\mathscr{D}$. Their commutation relations are

$$[P_a, P_b] = 0, \quad [P_a, M_b{}^c] = i\hbar(g_{ab}P^c - g_a{}^cP_b), \tag{1.1}$$

$$[M_a{}^b, M_c{}^d] = i\hbar(g_{ac}M^{db} - g_a{}^dM_c{}^b - g^{db}M_{ac} + g_c{}^bM_a{}^d), \tag{1.2}$$

$$[D, P_a] = -iP_a, \quad [D, M_b{}^c] = 0, \quad [D, D] = 0, \tag{1.3}$$

$$[P_a, Q^b] = -i\hbar(M_a{}^b - \hbar Dg_a{}^b), \tag{1.4}$$

$$[M_b{}^c, Q^a] = i\hbar(g^{ca}Q_b - g_b{}^aQ^c), \tag{1.5}$$

$$[D, Q^a] = iQ^a, \tag{1.6}$$

$$[Q^a, Q^b] = 0. \tag{1.7}$$

Equations (1.1) and (1.2) are those for $\mathscr{P}$; (1.1)–(1.3) are for $\mathscr{D}$. $\mathscr{C}$, $\mathscr{P}$ and $\mathscr{D}$ are real algebras. In (1.1)–(1.7) the commutators contain the factor i this being more appropriate for the discussion of representations in terms of hermitian operators.

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Let $\mathscr{C}_e$, $\mathscr{D}_e$ and $\mathscr{P}_e$ denote respectively the enveloping algebras of $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{P}$. Adjoin to each of $\mathscr{C}_e$, $\mathscr{D}_e$ and $\mathscr{P}_e$ the element m^{-2} and its powers m^{-2r}, where r is an integer, $r > 0$, obeying

$$m^{-2}P_a P^a = P_a P^a m^{-2} = 1, \tag{1.8}$$

$$[m^{-2}, X] = -m^{-2}[P_a P^a, X]\, m^{-2} \tag{1.9}$$

for any $X \in \mathscr{C}_e$. Call the resulting algebras $\mathscr{C}_e(m^{-2})$, $\mathscr{D}_e(m^{-2})$ and $\mathscr{P}_e(m^{-2})$. The element m^{-2} is then the inverse of $P_a P^a \equiv m^2$, the mass squared, and will often be written $1/m^2$.

Next define

$$M^{*ab} \equiv \tfrac{1}{2}\epsilon^{ab}{}_{cd}\, M^{cd}. \tag{1.10}$$

The following is the first main result:

Theorem A. *Define in* $\mathscr{C}_e(m^{-2})$, R^a *by the formula*

$$R^a \equiv Q^a - \tfrac{1}{4}M_{ef}\frac{P^a}{m^2}M^{ef} - M^{d(a}P_b/m^2\, M^{b)}{}_d - \tfrac{1}{2}\hbar\left(D\frac{P_b}{m^2}M^{ab} + M^{ab}\frac{P_b}{m^2}D\right) - \tfrac{1}{2}\hbar^2 D\frac{P^a}{m^2}D + 2\hbar^2\frac{P^a}{m^2}. \tag{1.11}$$

Then R^a *is manifestly hermitian if* Q^a, P_b *and* $M_a{}^b$ *are, and is translationally invariant:*

$$[P_b, R^a] = 0. \tag{1.12}$$

Also, $m^2R^a - m^2Q^a \in \mathscr{D}_e$, *and* R^a *has the following commutation relations:*

$$[D, R^a] = iR^a, \quad [M_b{}^c, R^a] = i\hbar(g^{ac}R_b - g_b{}^a R^c), \tag{1.13}$$

$$[R^a, R^b] = (2i\hbar/m^2)\,[R^cP_cM^{ab} + R^cM_c{}^{[a}P^{b]} + P^{[a}M^{b]}{}_c R^c]. \tag{1.14}$$

Finally, the freedom in choice of $R^a \in \mathscr{C}_e(m^{-2})$, *obeying*

$$[m^2, R^a] = 0, \quad m^2(R^a - Q^a) \in \mathscr{D}_e, \tag{1.15}$$

is $R^a \to R^a + m^{-2}g^a$ *where* $g^a \in \mathscr{P}_e$ *and* g^a *belongs to the sub-algebra of* $\mathscr{P}_e$ *generated by* P^a *and* $M^{*ab}P_b$.

Although one may prove theorem A by a direct computation, it is useful to develop some extra structure to render the basic definition (1.11) more transparent. First extend $\mathscr{C}$ trivially by an abelian one-real-dimensional algebra generated by an element β satisfying $[\beta, c] = 0$, for any $c \in \mathscr{C}_e(m^{-2})$, $[\beta, \beta] = 0$. Denote the resulting algebra $(U(2,2))$, by $\mathscr{U}$, with enveloping algebra $\mathscr{U}_e$ and m^{-2}-extension $\mathscr{U}_e(m^{-2})$.

Next define, in succession,

$$\lambda \equiv D - i\beta/2\hbar, \quad \bar{\lambda} \equiv D + i\beta/2\hbar; \tag{1.16}$$

the Pauli–Lubánski spin vector (in $\mathscr{P}_e$):

$$S_a \equiv M^*{}_a{}^b P_b = \tfrac{1}{2}\epsilon_a{}^{bcd}M_{bc}P_d; \tag{1.17}$$

the constrained relativistic position operator (in $\mathscr{P}_e(m^{-2})$):

$$X^a \equiv \tfrac{1}{2}m^{-2}(M^{ab}P_b - P_bM^{ba}); \tag{1.18}$$

the constrained complex position operator (in $\mathscr{P}_e(m^{-2})$):

$$Y^a \equiv X^a + im^{-2}S^a, \quad \bar{Y}^a \equiv X^a - im^{-2}S^a \tag{1.19}$$

(X^a, Y^a and $\bar{Y}^a$ obey the constraint identities $X^aP_a = Y^aP_a = \bar{Y}^aP_a = -\frac{3}{2}i\hbar = -P_aX^a = -P_aY^a = -P_a\bar{Y}^a$); the unconstrained position operators:

$$\tilde{X}^a \equiv X^a + \tfrac{1}{2}\hbar\left(D\frac{P^a}{m^2} + \frac{P^a}{m^2}D\right), \tag{1.20}$$

$$Z^a \equiv Y^a + \tfrac{1}{2}\hbar\left(\lambda\frac{P^a}{m^2} + \frac{P^a}{m^2}\lambda\right), \tag{1.21}$$

$$\bar{Z}^a \equiv \bar{Y}^a + \tfrac{1}{2}\hbar\left(\bar{\lambda}\frac{P^a}{m^2} + \frac{P^a}{m^2}\bar{\lambda}\right); \tag{1.22}$$

and the operator:

$$\begin{aligned}\hat{R}^a &\equiv Q^a - Z^{AB'}P_{B'B}\bar{Z}^{BA'} \\ &= Q^a - \tfrac{1}{2}(i\epsilon^a{}_{bcd}Z^bP^c\bar{Z}^d + Z^aP_c\bar{Z}^c + Z^cP_c\bar{Z}^a - Z^cP^a\bar{Z}_c).\end{aligned} \tag{1.23}$$

Then $[P_b, \hat{R}^a] = 0$, $m^2\hat{R}^a \in \mathscr{U}_e$ and

$$R^a = \hat{R}^a - \frac{1}{2m^2}\beta S^a + \frac{1}{8m^2}\beta^2P^a \tag{1.24}$$

$$= Q^a - Y^{AB'}P_{BB'}\bar{Y}^{BA'} - \tfrac{1}{2}\hbar(DX^a + X^aD) - \tfrac{1}{2}\hbar^2D\frac{P^a}{m^2}D + \tfrac{1}{8}\hbar^2\frac{P^a}{m^2}. \tag{1.25}$$

Theorem A now has the alternative form:

THEOREM A′. *The algebra $\mathscr{U}_e(m^{-2})$ may be generated by P_a, Z^a and $\hat{R}^a$ obeying the relations*

$$[P_a, P_b] = 0, \quad [P_a, Z^b] = [P_a, \bar{Z}^b] = i\hbar\delta_a{}^b, \tag{1.26}$$

$$[Z^a, Z^b] = 0, \quad [\bar{Z}^a, \bar{Z}^b] = 0, \quad [Z^a, \bar{Z}^b] = -(2i\hbar/m^2)(Z^{AB'} - \bar{Z}^{AB'})P^{BA'}, \tag{1.27}$$

$$[P_a, \hat{R}^b] = 0, \tag{1.28}$$

$$[Z^a, \hat{R}^b] = (2i\hbar/m^2)\hat{R}^{AB'}P^{BA'}, \quad [\bar{Z}^a, \hat{R}^b] = (2i\hbar/m^2)\hat{R}^{BA'}P^{AB'}, \tag{1.29}$$

$$[\hat{R}^a, \hat{R}^b] = -i\hbar[\hat{R}^{BA'}(Z^{AB'} - \bar{Z}^{AB'}) - (Z^{BA'} - \bar{Z}^{BA'})\hat{R}^{AB'}], \tag{1.30}$$

with the relations

$$M^{ab} = \epsilon^{A'B'}Z^{(A}{}_{C'}P^{B)C'} + \epsilon^{AB}\bar{Z}^{(A'}{}_CP^{B')C}, \tag{1.31}$$

$$Q^a = \hat{R}^a + Z^{AB'}P_{BB'}\bar{Z}^{BA'}, \tag{1.32}$$

$$D = (1/2\hbar)(P_aZ^a + \bar{Z}^aP_a), \quad \beta = i(P_aZ^a - P_a\bar{Z}^a), \tag{1.33}$$

$$\tilde{X}^a = \tfrac{1}{2}(Z^a + \bar{Z}^a), \quad S^a = \tfrac{1}{2}(m^2/i)(Z^a - \bar{Z}^a) + \tfrac{1}{2}\beta P^a, \tag{1.34}$$

$$Y^a = Z^a - \frac{1}{2}\left(Z^bP_b\frac{P^a}{m^2} + \frac{P^a}{m^2}P_bZ^b\right), \tag{1.35}$$

$$\bar{Y}^a = \bar{Z}^a - \frac{1}{2}\left(\bar{Z}^bP_b\frac{P^a}{m^2} + \frac{P^a}{m^2}P_b\bar{Z}^b\right). \tag{1.36}$$

Again one proves theorem A′ by direct computations, simpler though than those involved in theorem A.

Thus P_a, Z^a and $\hat{R}^a$ completely characterize the structure of $\mathscr{U}_e(m^{-2})$, with Z^a complex. P_a and Z^a together describe the enveloping algebra $\mathscr{D}_e(\beta)$ of the extension $\mathscr{D}(\beta)$ of $\mathscr{D}$ by the abelian piece β.

COROLLARY A 1. *Let $\mathscr{R}$ denote the subalgebra of $\mathscr{C}_e(m^{-2})$ defined by*

$$\mathscr{R} \equiv \{r \in \mathscr{C}_e(m^{-2}); \quad [m^2, r] = 0\}. \tag{1.37}$$

Then $\mathcal{R}$ is generated by R^a, P_b and $M_a{}^b$. Also $\mathscr{C}_e(m^{-2})$ is then the extension of $\mathcal{R}$, $\mathcal{R}(D)$ obtained by adding the single extra generator D obeying

$$[D,D] = 0, \quad [D,P^a] = iP^a, \quad [D,R_a] = -iR_a, \quad [D,M_a{}^b] = 0. \tag{1.38}$$

That $\mathcal{R}$ is a subalgebra follows from its definition by the use of the Jacobi identity. That $\mathcal{R}$ is generated by R^a, P_b and $M_a{}^b$ follows immediately from theorem A by inspection.

Next the Casimir operators of the Poincaré subalgebra $\mathscr{P}$ are $P_aP^a \equiv m^2$, the mass squared, and $\vec{J}^2 \equiv -m^{-2}S^aS_a$, the relativistic spin squared. These obey:

$$[m^2, R^a] = 0, \quad [m^2, D] = 2im^2, \tag{1.39}$$

$$[\vec{J}^2, D] = 0, \quad [\vec{J}^2, R^a] = 2i\hbar m^{-2}\epsilon^{abcd}P_bR_cS_d + 2\hbar^2R^a - 2\hbar^2R^cP_cP^a. \tag{1.40}$$

Corollary A 2. *Define*

$$C_2 \equiv R^aP_a + \vec{J}^2, \tag{1.41}$$

$$C_3 \equiv R^aS_a, \tag{1.42}$$

$$C_4 \equiv m^2R^aR_a + 2\hbar^2\vec{J}^2. \tag{1.43}$$

Then C_2, C_3 and $C_4 \in \mathscr{C}_e$, and are the three Casimir operators of $\mathscr{C}_e$.

Again the proof is by direct computation. Corollary A 2 has the following alternative form:

Corollary A 2′. *Put*

$$\hat{C}_1 \equiv \beta, \tag{1.44}$$

$$\hat{C}_2 \equiv \hat{R}^aP_a + \vec{J}^2, \tag{1.45}$$

$$\hat{C}_3 \equiv \hat{R}^aS_a + \tfrac{1}{2}\beta\vec{J}^2, \tag{1.46}$$

$$\hat{C}_4 \equiv m^2\hat{R}^a\hat{R}_a + 2\hbar^2\vec{J}^2. \tag{1.47}$$

Then $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3$ and $\hat{C}_4 \in \mathscr{U}_e$, and are the four Casimir operators of $\mathscr{U}_e$. Also

$$C_2 = \hat{C}_2 + \tfrac{1}{8}\beta^2, \tag{1.48}$$

$$C_3 = \hat{C}_3, \tag{1.49}$$

$$C_4 = \hat{C}_4 - \beta\hat{C}_3 + \tfrac{1}{4}\beta^2\hat{C}_2 + \tfrac{1}{64}\beta^4. \tag{1.50}$$

Corollary A 3. *The Casimir operators of $\mathcal{R}$ are C_2, C_3, C_4 and m^2.*

The behaviour of the various operators constructed from the conformal algebra under discrete symmetries may be inferred from their realization as generators of symmetries of compacted space–time. One finds that under a charge-parity inversion denoted by $\mathcal{cp}$, the quantities P^a, M^{ab}, Q^a, D transform according to

$$\mathcal{cp}P_a(\mathcal{cp})^{-1} = t_A^{B'}P_bt_{A'}^B, \tag{1.51}$$

$$\mathcal{cp}M^{ab}(\mathcal{cp})^{-1} = t_{C'}^At_C^{A'}t_{D'}^Bt_D^{B'}M^{cd}, \tag{1.52}$$

$$\mathcal{cp}Q^a(\mathcal{cp})^{-1} = t_{B'}^At_B^{A'}Q^b, \tag{1.53}$$

$$\mathcal{cp}D(\mathcal{cp})^{-1} = D, \tag{1.54}$$

where $t^at_a = 2$ and the spatial inversion leaves t_a invariant. These transformations induce

$$\mathcal{cp}M^{*ab}(\mathcal{cp})^{-1} = -t_{C'}^At_C^{A'}t_{D'}^Bt_D^{B'}M^{*cd}, \tag{1.55}$$

$$\mathcal{cp}S^a(\mathcal{cp})^{-1} = -t_{B'}^At_B^{A'}S^b, \tag{1.56}$$

$$\mathcal{cp}X^a(\mathcal{cp})^{-1} = t_{B'}^At_B^{A'}X^b, \tag{1.57}$$

$$\mathcal{cp}Y^a(\mathcal{cp})^{-1} = t_{B'}^At_B^{A'}\bar{Y}^b, \tag{1.58}$$

$$\mathcal{cp}R^a(\mathcal{cp})^{-1} = t_{B'}^At_B^{A'}R^b. \tag{1.59}$$

Taking in addition
$$c p \beta (c p)^{-1} = -\beta, \tag{1.60}$$
one has
$$c p \lambda (c p)^{-1} = \bar{\lambda}, \tag{1.61}$$
$$c p Z^a (c p)^{-1} = t^A_{B'} t^{A'}_B \bar{Z}^b, \tag{1.62}$$
$$c p \hat{R}^a (c p)^{-1} = t^A_{B'} t^{A'}_B \hat{R}^b, \tag{1.63}$$
$$c p m^2 (c p)^{-1} = m^2, \tag{1.64}$$
$$c p \vec{J}^2 (c p)^{-1} = \vec{J}^2, \tag{1.65}$$
$$c p C_2 (c p)^{-1} = C_2, \tag{1.66}$$
$$c p C_3 (c p)^{-1} = -C_3, \tag{1.67}$$
$$c p C_4 (c p)^{-1} = C_4, \tag{1.68}$$
$$c p \hat{C}_2 (c p)^{-1} = \hat{C}_2, \tag{1.69}$$
$$c p \hat{C}_3 (c p)^{-1} = -\hat{C}_3, \tag{1.70}$$
$$c p \hat{C}_4 (c p)^{-1} = \hat{C}_4. \tag{1.71}$$

The charge parity inversion gives an automorphism for each of the algebras under discussion, with square the identity transformation.

Geometrically, the transformation $c p$ follows from the space–time parity transformation. Why, then, is it not the operator corresponding to parity?

Because, from (1.67), $c p$ changes the value of a Casimir operator. Thus the nature of the system is changed by the transformation. It cannot be the parity transformation. What, then, is the parity transformation p? Or, equivalently, what is c? c must be an automorphism leaving P^a, M^{ab} and D invariant which changes back the sign of C_3. Surprisingly, such an automorphism does exist, satisfying
$$c P^a c^{-1} = P^a, \tag{1.72}$$
$$c M^{ab} c^{-1} = M^{ab}, \tag{1.73}$$
$$c D c^{-1} = D, \tag{1.74}$$
$$\begin{aligned} c R^a c^{-1} &= -R^a + 2R^b P_b (P^c P_c)^{-1} P^a \\ &= 2P^A_{B'} R^b P^{A'}_B (P^c P_c)^{-1}. \end{aligned} \tag{1.75}$$

c is an automorphism which does not come from the Weyl symmetries of the conformal algebra, although it is a symmetry of the extended enveloping algebra $\mathscr{C}_e(m^{-2})$.

If one takes in addition
$$c \beta c^{-1} = -\beta, \tag{1.76}$$
then one has
$$c \hat{R}^a c^{-1} = -\hat{R}^a + 2\hat{R}^b P_b (P^c P_c)^{-1} P^a, \tag{1.77}$$
$$c \lambda c^{-1} = \bar{\lambda}. \tag{1.78}$$
Also
$$c M^{*ab} c^{-1} = M^{*ab}, \tag{1.79}$$
$$c S^a c^{-1} = S^a, \tag{1.80}$$
$$c X^a c^{-1} = X^a, \tag{1.81}$$
$$c Y^a c^{-1} = Y^a, \tag{1.82}$$

$$\mathscr{c}Z^a\mathscr{c}^{-1} = Z^a + \tfrac{1}{2}i\beta P^a(P^cP_c)^{-1}, \tag{1.83}$$

$$\mathscr{c}m^2\mathscr{c}^{-1} = m^2, \tag{1.84}$$

$$\mathscr{c}\vec{J}^2\mathscr{c}^{-1} = \vec{J}^2, \tag{1.85}$$

$$\mathscr{c}C_2\mathscr{c}^{-1} = C_2, \tag{1.86}$$

$$\mathscr{c}C_3\mathscr{c}^{-1} = -C_3, \tag{1.87}$$

$$\mathscr{c}C_4\mathscr{c}^{-1} = C_4, \tag{1.88}$$

$$\mathscr{c}\hat{C}_2\mathscr{c}^{-1} = \hat{C}_2, \tag{1.89}$$

$$\mathscr{c}\hat{C}_3\mathscr{c}^{-1} = -\hat{C}_3, \tag{1.90}$$

$$\mathscr{c}\hat{C}_4\mathscr{c}^{-1} = \hat{C}_4, \tag{1.91}$$

One has $\mathscr{c}\mathscr{p} = \mathscr{p}\mathscr{c}$. The parity transformation $\mathscr{p}$ is then the product $\mathscr{c}\mathscr{p}\mathscr{c}$. $\mathscr{p}$ agrees with $\mathscr{c}\mathscr{p}$ on the algebra $\mathscr{D}_e(m^{-2})$ and transforms R^a according to

$$\mathscr{p}R^a\mathscr{p}^{-1} = 2(P^eP_e)^{-1}\, t_C^{A'} P_{B'}^{C} R^b P_B^{C'} t_{C'}^{A}. \tag{1.92}$$

The transformation $\mathscr{p}$ leaves invariant m^2, $\vec{J}^2$, C_2, C_3 and C_4.

In a momentum state, one may regard P^a as a c-number. Then t^a may be taken proportional to P^a (this is equivalent to going to the rest frame). In this case (1.92) boils down to just

$$\mathscr{p}R^a\mathscr{p}^{-1} = R^a. \tag{1.93}$$

So the R^a operator preserves parity. All the states in a fixed $\mathscr{R}$-algebra representation must have the same parity. At the same time their behaviour under $\mathscr{c}\mathscr{p}$ must alternate with spin. These are very stringent requirements on any attempt at a physical interpretation of the conformal properties of the system.

2. Representations of the $\mathscr{R}$-algebra

The strategy is to combine suitable Poincaré representations to build an Hermitian representation of the $\mathscr{R}$-algebra. The mass is fixed and positive throughout. Since $[P_a, R^b] = 0$, a 'little algebra' approach may be used and the entire discussion confined to a fixed momentum. So only the spin parts of the wave function need to be considered. Thus the problem is reduced to finding Hermitian representations of the little algebra:

$$[R^a, P_b] = [S^a, P_b] = [P^a, P_b] = 0, \tag{2.1}$$

$$[R^a, S^b] = -i\hbar\epsilon^{abcd}R_cP_d, \tag{2.2}$$

$$[S^a, S^b] = -i\hbar\epsilon^{abcd}S_cP_d, \tag{2.3}$$

$$\begin{aligned}[R^a, R^b] &= i\hbar m^{-2}\epsilon^{abcd}(R_cS_d - S_cR_d)\\ &= 2i\hbar m^{-2}\epsilon^{abcd}R_cS_d + 2\hbar^2 m^{-2}(R^aP^b - R^bP^a).\end{aligned} \tag{2.4}$$

R^a, S^a and P^a act on states $|j, q\rangle$, which may be taken to be eigenstates of P^a with eigenvalue also written P^a, of relativistic spin $j\hbar$. They are acted on by the spin vector S^a in the standard Poincaré fashion:

$$\begin{aligned}S^a|j, q\rangle = \hbar(\bar{\alpha}_D\alpha_{D'}P^{DD'})^{-1}[&\alpha^{A'}\alpha^{B'}P_{B'}^{A} j, q+1\rangle + (j-q)\,(\bar{\alpha}^A\alpha^{A'} - \alpha^{B'}P_{B'}{}^{A}\bar{\alpha}^BP_B{}^{A'})|j, q\rangle\\ &+ q(2j-q+1)\,\bar{\alpha}^A\alpha^BP_B{}^{A'}|j, q-1\rangle].\end{aligned} \tag{2.5}$$

In particular, they satisfy

$$S^{AA'}\bar{\alpha}_A\,\alpha_{A'}\,|j,q\rangle = -P^{AA'}\bar{\alpha}_A\,\alpha_{A'}\,\hbar(j-q)\,|j,q\rangle, \tag{2.6}$$

where q and $2j$ are integers satisfying $0 \leqslant q \leqslant 2j$ (here and in the following P^aP_a has been normalized to the value 2; for un-normalized P^a, simply replace P^a everywhere by $(\frac{1}{2}m^2)^{-\frac{1}{2}}P^a$). From the work of Mack (1977), one may take the spin spectrum to be non-degenerate.

The state $|j,q\rangle$ is orthogonal to $|j',q'\rangle$ unless $j=j'$ and $q=q'$, and has normalization

$$\langle j,q\,|\,j,q\rangle = d_j\,q!(\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^{2j}\,[(2j+1)\,(2j-q)!]^{-1}, \tag{2.7}$$

which makes S^a Hermitian. The $d_j > 0$ may be chosen at one's convenience.

On the basis of general considerations for the vector–scalar operator, R^a, one must have

$$R^a\,|j,q\rangle = \hbar^2\alpha_j\,P^a\,|j,q\rangle + \hbar\beta_j S^a\,|j,q\rangle + \hbar^2\gamma_j[(2j+1)\,(2j+2)\,(\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^2]^{-1}\,L^a\,|j,q\rangle + \hbar^2\delta_j\,\tilde{L}^a\,|j,q\rangle, \tag{2.8}$$

where L^a and $\tilde{L}^a$ are, respectively, generic spin-raising and lowering operators defined by

$$L^a\,|j,q\rangle = -\alpha^{A'}\alpha^{B'}P_{B'}{}^{A}|j+1,q+2\rangle - (2j+1-q)\,(\bar{\alpha}^A\alpha^{A'} - \alpha^{B'}P_{B'}{}^{A}\bar{\alpha}^B P_B{}^{A'})\,|j+1,q+1\rangle + (2j+1-q)\,(2j+2-q)\,\bar{\alpha}^A\bar{\alpha}^B P_B{}^{A'}\,|j+1,q\rangle, \tag{2.9}$$

$$\tilde{L}^a\,|j,q\rangle = \alpha^{A'}\alpha^{B'}P_{B'}{}^{A}\,|j-1,q\rangle - q(\bar{\alpha}^A\alpha^{A'} - \alpha^{B'}P_{B'}{}^{A}\bar{\alpha}^B P_B{}^{A'})\,|j-1,q-1\rangle - q(q-1)\,\bar{\alpha}^A\bar{\alpha}^B P_B{}^{A'}\,|j-1,q-2\rangle, \tag{2.10}$$

and where α_j, β_j, γ_j and δ_j are appropriate Clebsch–Gordan coefficients to be determined (when $j=0$ occurs in the representation put $\beta_0 = 0$).

For an irreducible representation, one may assume that the spins j form an unbroken string: all j satisfying $j = j_{\min}+n$ for some fixed $j_{\min}$ and either for all non-negative integers n or for all integers n satisfying $0 \leqslant n \leqslant j_{\max}-j_{\min}$, for some fixed $j_{\max}$ with $j_{\max}-j_{\min}$ a non-negative integer. Then $\delta_{j_{\min}} = 0$ and one can define $\delta_j = \gamma_{j-1} = 0$ for $j \leqslant j_{\min}$. Also if $j_{\max}$ exists, $\gamma_{j_{\max}} = 0$ and one can define $\delta_{j+1} = \gamma_j = 0$ for $j \geqslant j_{\max}$. Next define

$$m^a \equiv \alpha^{A'}\alpha^{B'}P_{B'}{}^{A}, \quad \bar{m}^a \equiv \bar{\alpha}^A\bar{\alpha}^B P_B{}^{A'}, \quad l^a \equiv \bar{\alpha}^A\alpha^{A'} - \alpha^{B'}P_{B'}{}^{A}\,\bar{\alpha}^B P_B{}^{A'}. \tag{2.11}$$

One has
$$P^a(\bar{\alpha}^D P_{DD'}\,\alpha^{D'}) = \bar{\alpha}^A\alpha^{A'} + \bar{\alpha}^B P_B{}^{A'}\alpha^{B'}P_{B'}{}^{A},$$

$$m^a\bar{m}_a = \tfrac{1}{2}l^a l_a = -(\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^2, \tag{2.12}$$

$$m^a m_a = \bar{m}^a\bar{m}_a = m^a P_a = \bar{m}^a P_a = m^a l_a = \bar{m}^a l_a = l^a P_a = 0, \tag{2.13}$$

$$2m^{[a}l^{b]} = (\alpha^{A'}\alpha^{B'}\epsilon^{AB} + \alpha^{C'}P_{C'}{}^{A}\alpha^{D'}P_{D'}{}^{B}\epsilon^{A'B'})\,(\bar{\alpha}^E P_{EE'}\,\alpha^{E'}), \tag{2.14}$$

$$2m^{[a}P^{b]} = \alpha^{A'}\alpha^{B'}\epsilon^{AB} - \alpha^{C'}P_{C'}{}^{(A}\,\bar{\alpha}^{B)}\epsilon^{A'B'}, \tag{2.15}$$

$$2m^{[a}\bar{m}^{b]} = (\bar{\alpha}^C P_C{}^{(A'}\alpha^{B')}\epsilon^{AB} - \alpha^{C'}P_{C'}{}^{(A}\bar{\alpha}^{B)}\epsilon^{A'B'})\,(\bar{\alpha}^D P_{DD'}\,\alpha^{D'}), \tag{2.16}$$

$$2l^{[a}P^{b]} = -2\bar{\alpha}^C P_C{}^{(A'}\alpha^{B')}\epsilon^{AB} - 2\alpha^{C'}P_{C'}{}^{(A}\bar{\alpha}^{B)}\epsilon^{A'B'}, \tag{2.17}$$

$$\epsilon^{ab}{}_{cd}\,m^c l^d = 2im^{[a}P^{b]}(\bar{\alpha}^D P_{DD'}\,\alpha^{D'}), \tag{2.18}$$

$$\epsilon^{ab}{}_{cd}\,m^c P^d = 2im^{[a}l^{b]}(\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^{-1}, \tag{2.19}$$

$$\epsilon^{ab}{}_{cd}\,m^c\bar{m}^d = il^{[a}P^{b]}(\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^{-1}, \tag{2.20}$$

$$\epsilon^{ab}{}_{cd}\,l^c P^d = -4im^{[a}\bar{m}^{b]}(\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^{-1}, \tag{2.21}$$

$$\epsilon_{abcd} = 12il_{[a}m_b\bar{m}_c P_{d]}(\bar{\alpha}^E P_{EE'}\,\alpha^{E'})^{-3}. \tag{2.22}$$

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Also let J, Q, $\alpha(J)$, $\beta(J)$, $\gamma(J)$ and $\delta(J)$ be the operators whose eigenvalues acting on $|j,q\rangle$ are respectively j, q, α_j, β_j, γ_j and δ_j. Then

$$R^a = \hbar^2\alpha(J)\,P^a + \hbar\beta(J)\,S^a + \hbar^2 L^a[(2J+1)\,(2J+2)\,(\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^2]^{-1}\,\gamma(J) + \hbar\tilde{L}^a(2J+1)^{-1}\,\delta(J) \tag{2.23}$$

and the expressions for S^a, L^a, $\tilde{L}^a$ may be rewritten

$$S^a\,|j,q\rangle = (\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^{-1}\,\hbar[m^a\,|j,q+1\rangle + (j-q)\,l^a\,|j,q\rangle + q(2j-q+1)\,\bar{m}^a\,|j,q-1\rangle], \tag{2.24}$$

$$\begin{aligned} L^a\,|j,q\rangle = &-m^a\,|j+1,q+2\rangle - (2j+1-q)\,l^a\,|j+1,q+1\rangle \\ &+ (2j+1-q)\,(2j+2-q)\,\bar{m}^a\,|j+1,q\rangle, \end{aligned} \tag{2.25}$$

$$\tilde{L}^a\,|j,q\rangle = \tilde{m}^a\,|j-1,q\rangle - ql^a\,|j-1,q-1\rangle - q(q-1)\,\bar{m}^a\,|j-1,q-2\rangle. \tag{2.26}$$

One sees that

$$\begin{aligned} P^aS_a = P^aL_a = P^a\tilde{L}_a = S^aL_a = S^a\tilde{L}_a = L^aL_a = \tilde{L}^a\tilde{L}_a \\ = S^aP_a = L_aP^a = \tilde{L}_aP^a = L_aS^a = \tilde{L}_aS^a = 0. \end{aligned} \tag{2.27}$$

Then, contracting (2.23) with P_a and S_a, one finds using (1.41), (1.42):

$$2\alpha(J) = \hbar^{-2}C_2 - J(J+1), \tag{2.28}$$

$$2J\,(J+1)\,\beta(J) = -\hbar^{-3}C_3, \tag{2.29}$$

and $\beta(J) = C_3 = 0$ if $j = 0$ occurs in the representation. Next

$$L^a\tilde{L}_a = -2(J+1)\,(2J+3)\,(\bar{\alpha}^D P_{DD'}\alpha^{D'})^2, \tag{2.30}$$

$$\tilde{L}^aL_a = -2J(2J-1)\,(\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^2. \tag{2.31}$$

Hence, by using (1.43) and (2.23)

$$\begin{aligned} \hbar^{-4}C_4 = &\,2J\,(J+1) + 4\alpha^2(J) - 4J\,(J+1)\,\beta^2(J) \\ &-2\delta(J+1)\,\gamma(J)\,(2J+1)^{-1} - 2\delta(J)\,\gamma(J-1)\,(2J+1)^{-1}. \end{aligned} \tag{2.32}$$

From (2.28), (2.29), (2.32) one obtains

$$\begin{aligned} 2\delta_{j+1}\gamma_j + 2\delta_j\gamma_{j-1} = &\,(2j+1)j(j+1)\,(j^2+j+2) - (2j+1)\,[j(j+1)]^{-1} \\ &+ (2j+1)\,(c_2^2 - c_4) - 2(2j+1)j(j+1)\,c_2, \end{aligned} \tag{2.33}$$

where c_2, c_3 and c_4 are, respectively, the eigenvalues of $\hbar^{-2}C_2$, $\hbar^{-3}C_3$, and $\hbar^{-4}C_4$. If $j = 0$ occurs, omit the term containing c_3 in (2.33). Then (2.33) applies for *each* j occurring in the representation.

Now define k_j for each $j \geqslant j_{\min} - 1$ by

$$2\delta_{j+1}\gamma_j \equiv (c_2^2 - c_4)\,(j+1) - 2c_2j(j+1)\,(j+2) + j(j+1)^3\,(j+2) - c_3^2(j+1)^{-1} + (-)^j\,k_j \tag{2.34}$$

if $j_{\min} \neq 0$. If $j_{\min} = 0$, omit the term containing c_3. Then substituting into (2.33) gives, for each j occurring in the representation, $k_j = k_{j-1}$ so $k_j = k$ is independent of j for $j = j_{\min} - 1$ and for all j in the representation.

To find k, consider (2.4). Noting that $[L^a, L^b] = [\tilde{L}^a, \tilde{L}^b] = 0$, one sees that $[R^a, R^b]$ changes spin by 1, 0 or -1 unit. Examination shows that the spin-changing pieces give no new information. So it is sufficient to look at the terms of unchanged j in $2R^{[a}R^{b]}\,|j,q\rangle$. Now

$$2\tilde{L}^{[a}L^{b]} = i\hbar^{-1}(2J+3)\,(\bar{\alpha}^E P_{EE'}\,\alpha^{E'})^2\,\epsilon^{abcd}S_c\,P_d, \tag{2.35}$$

$$2L^{[a}\tilde{L}^{b]} = -i\hbar^{-1}(2J+1)\,(\bar{\alpha}^E P_{EE'}\,\alpha^{E'})^2\,\epsilon^{abcd}S_c\,P_d. \tag{2.36}$$

So using (2.35), (2.36), (2.1), (2.3), and (2.23) one has, modulo spin-changing pieces,

$$2R^{[a}R^{b]} = i\hbar^3[-\beta^2(J)+\delta(J+1)\,\gamma(J)\{(2J+1)\,(2J+2)\}^{-1}$$
$$-\delta(J)\,\gamma(J+1)\,\{2J(2J+1)\}^{-1}]\,\epsilon^{abcd}S_c\,P_d \tag{2.37}$$

(omit the $\delta(J)$ term, when $J=0$).

The terms of unchanged spin in the right-hand side of (2.4) are

$$i\hbar\epsilon^{abcd}(\hbar^2\alpha(J)\,P_c+\hbar\beta(J)\,S_c)\,S_d+2\hbar^3\beta(J)\,S^{[a}P^{b]} = -i\hbar^3\epsilon^{abcd}S_c\,P_d\,\alpha(J). \tag{2.38}$$

Hence equating (2.37) and (2.38), one must have, when $j \neq 0$, but otherwise for each j occurring in the representation,

$$\delta_{j+1}\,\gamma_j[(2j+1)\,(2j+2)]^{-1}-\delta_j\,\gamma_{j-1}[2j(2j+1)]^{-1} = \beta_j^2+\alpha_j. \tag{2.39}$$

When $j=0$ there is no condition, as (2.37) and (2.38) both vanish.

Now define l_j, for each j occurring in the representation and for $j = j_{\min}-1$, when $j_{\min} \neq 0$, by

$$\delta_{j+1}\,\gamma_j(2j+2) = -c_3^2[4(j+1)^2]^{-1}+\tfrac{1}{4}(j+1)^4-\tfrac{1}{2}(c_2+\tfrac{1}{2})\,(j+1)^2+\tfrac{1}{4}l_j. \tag{2.40}$$

Then substituting (2.40) into (2.39), one finds $l_j = l_{j-1} = l$ is independent of j. So for $j \geqslant j_{\min}-1$, $(j \neq -1)$,

$$2\delta_{j+1}\,\gamma_j = -c_3^2(j+1)^{-1}+j(j+1)^3\,(j+2)-2c_2j(j+1)^2\,(j+2)+(j+1)\,(l-2c_2). \tag{2.41}$$

Comparing with (2.34), one obtains, provided zero is not the only spin occurring in the representation,

$$k = 0, \tag{2.42}$$

$$l = 2c_2+c_2^2-c_4. \tag{2.43}$$

So, finally, to summarize: for $j = j_{\min}-1$ and for all j occurring in the representation, provided only that $j=0$ is not the only spin occurring in the representation, one finds

$$2\delta_{j+1}\,\gamma_j = (c_2^2-c_4)\,(j+1)+j(j+1)\,(j+2)\,[(j+1)^2-2c_2]-c_3^2(j+1)^{-1}, \tag{2.44}$$

where for $j=-1$, the last term is omitted. If $j=0$ is the only spin occurring in the representation, then (2.32) reduces to $c_4 = c_2^2$, since all δ_j and γ_j vanish. So (2.44) holds for all j in the representation and for $j = j_{\min}-1$, even when $j=0$ is the only spin.

Multiplying (2.44) by $j+1$ one defines $f(j)$ by

$$f(j) \equiv 2(j+1)\,\delta_{j+1}\,\gamma_j = (c_2^2-c_4)\,(j+1)^2+j(j+1)^2\,(j+2)\,((j+1)^2-2c_2)-c_3^2$$
$$= (j+1)^6-(j+1)^4\,(2c_2+1)-(j+1)^2\,(c_4-c_2^2-2c_2)-c_3^2. \tag{2.45}$$

Again, since $j=-1$ only occurs if $j=0$ does also, when $c_3=0$, (2.45) is valid for $j = j_{\min}-1$ and for all j occurring in the representation. The key point now is that $\gamma_{j_{\min}-1} = \delta_{j_{\min}} = 0$, and if $j_{\max}$ exists $\delta_{j_{\max}+1} = \gamma_{j_{\max}} = 0$, so that from (2.45), $f(j)$ has $j = j_{\min}-1$ and $j = j_{\max}$ (if it exists) as a root (possibly repeated).

Conversely, if α_j, β_j and $\delta_{j+1}\gamma_j$ are defined by (2.28), (2.29) and (2.44), together with $\delta_{j_{\min}} = 0$ and $\gamma_{j_{\max}} = 0$ (if a maximum spin exists), for complex numbers c_2, c_3, and c_4 satisfying $f(j_{\min}-1) = 0$ and $f(j_{\max}) = 0$ (if $j_{\max}$ exists), then S^a and R^a defined on Poincaré representations of fixed mass according to (2.5) and (2.8) give a not necessarily Hermitian representation of the $\mathscr{R}$-algebra, the Poincaré sub-algebra having an Hermitian representation. Each spin j occurs once in the range $j_{\min} \leqslant j \leqslant j_{\max}$ if $j_{\max}$ exists, and once in the range $j \geqslant j_{\min}$ if $j_{\max}$ does not exist, where $j-j_{\min}$ is integral, as is $j_{\max}-j_{\min}$.

If $f(j)$ is non-zero for $j_{\min} \leqslant j < j_{\max}$, $j-j_{\min}$ integral, then the representation is irreducible and has no sub-representations. If $f(j_0)$ vanishes with $j_{\min} \leqslant j_0 < j_{\max}$, $j_0-j_{\min}$ integral, and if $\delta_{j_0+1} = \gamma_{j_0} = 0$, then the representation is the direct sum of two irreducible representations, each with no sub-representations.

If $\gamma_{j_0} = 0$ but $\delta_{j_0+1} \neq 0$, then the states of spins j, $j_{\min} \leqslant j \leqslant j_0$, $j-j_{\min}$ integral, form a sub-representation, but the representation is not reducible to a direct sum of sub-representations. Similarly if $\gamma_{j_0} \neq 0$ but $\delta_{j_0+1} = 0$, then the states of spins j, $j_0 < j \leqslant j_{\max}$ from a sub-representation, but again the representation is not reducible to a direct sum of sub-representations.

One is fortunate that some of these subtleties are removed by requiring that the entire $\mathscr{R}$-representation be Hermitian. R^a will be Hermitian if and only if, for all j', q', j, q,

$$\langle j', q' | R^a | j, q \rangle = \overline{\langle j, q | R^a | j', q' \rangle}. \tag{2.46}$$

Using (2.5), (2.7), and (2.8), (2.46) boils down to

$$d_j \alpha_j = \bar{d}_j \bar{\alpha}_j, \tag{2.47}$$

$$d_j \beta_j = \bar{d}_j \bar{\beta}_j, \tag{2.48}$$

$$d_{j+1} \gamma_j = \bar{d}_j \bar{\delta}_{j+1} (2j+2). \tag{2.49}$$

Since $d_j = \bar{d}_j > 0$, one has

$$\alpha_j = \bar{\alpha}_j, \quad \beta_j = \bar{\beta}_j, \tag{2.50}$$

$$(2j+2)\,\delta_{j+1} = d_j^{-1} d_{j+1} \bar{\gamma}_j. \tag{2.51}$$

Also c_2, c_3 and c_4 are real, and from (2.45), (2.51)

$$0 \leqslant f(j) = d_j^{-1} d_{j+1} |\gamma_j|^2. \tag{2.52}$$

If the representation is irreducible and has no maximum spin, then $j_{\min}-1$ is the largest real root of $f(j) = 0$. In this case $f(j) = 0$ need have no other real roots than $j = -1 \pm j_{\min}$.

Now suppose the representation has a maximum spin. First consider the special case $j_{\max} = j_{\min} = j_0$, say, when only the spin $j = j_0$ occurs.

If $j_0 = 0$, R^a reduces to

$$R^a = \hbar^2 \alpha_0 P^a \tag{2.53}$$

with α_0 real. Then

$$c_2 = 2\alpha_0, \quad c_3 = 0, \quad c_4 = 4\alpha_0^2, \tag{2.54}$$

$$f(j) = (j+1)^2 [(j+1)^2 - 1][(j+1)^2 - 4\alpha_0]. \tag{2.55}$$

If $j_0 > 0$, R^a is given by

$$R^a = \hbar^2 \alpha_{j_0} P^a + \beta_{j_0} S^a, \tag{2.56}$$

where $\beta_{j_0}^2 = \alpha_{j_0}$, and β_{j_0} and α_{j_0} are real, $\alpha_{j_0} \geqslant 0$. Then

$$c_2 = 2\beta_{j_0}^2 + j_0(j_0+1), \tag{2.57}$$

$$c_3 = -2\beta_{j_0} j_0 (j_0+1), \tag{2.58}$$

$$c_4 = 2j_0(j_0+1) + 4\beta_{j_0}^4 - 4\beta_{j_0}^2 j_0 (j_0+1), \tag{2.59}$$

$$f(j) = [(j+1)^2 - j_0^2][(j+1)^2 - (j_0+1)^2][(j+1)^2 - 4\beta_{j_0}^2]. \tag{2.60}$$

Last consider the generic case of bounded spin, $j_{\max} > j_{\min}$. Then $f(j_{\min}-1) = f(j_{\max}) = 0$ and $f(j_{\min}+n) > 0$, for $0 \leqslant n < j_{\max} - j_{\min}$, n integral, and

$$f(j) = [(j+1)^2 - j_{\min}^2][(j+1)^2 - (j_{\max}+1)^2][(j+1)^2 - \alpha^2]. \tag{2.61}$$

Since $f(j_{\max}-1) > 0$, one infers that $\alpha^2 > j_{\max}^2$, so α may be taken real and $\alpha > j_{\max}$. Also with $\alpha > j_{\max}$, all conditions are satisfied.

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To tie down this case completely, choose d_j, after rescaling the states $|j, q\rangle$ by a real factor if necessary, to satisfy, for $j_{\min} \leqslant j \leqslant j_{\max} - 1$,

$$d_j^{-1} d_{j+1} = [j+1-(j_{\max}+1)]^{-1} (j+1-\alpha)^{-1} (j+1+j_{\max}+1)(j+1+\alpha) \times (j+1+j_{\min})(j+1-j_{\min}). \quad (2.62)$$

Then γ_j satisfies, for all j, $j_{\min} \leqslant j \leqslant j_{\max}$,

$$|\gamma_j|^2 = (j+1-\alpha)^2 (j-j_{\max})^2. \quad (2.63)$$

By suitably adjusting the phases of the states $|j, q\rangle$, if necessary, one may take γ_j real and non-negative, so

$$\gamma_j = (j+1-\alpha)(j-j_{\max}), \quad (2.64)$$

$$2j\delta_j = (j+\alpha)(j+j_{\max}+1)(j-j_{\min})(j+j_{\min}). \quad (2.65)$$

This completes the abstract description of the Hermitian positive-mass irreducible representations of the $\mathscr{R}$-algebra.

Summarizing the case of bounded spin, R^a is given by

$$R^a = K^a + \bar{K}^a + (C_2 - \vec{J}^2)(P^b P_b)^{-1} P^a - C_3 S^a (\vec{J}^2 P^b P_b)^{-1} \quad (2.66)$$

(with the last term omitted if a spin zero state occurs), where K^a and its adjoint $\bar{K}^a$ are defined by

$$K^a \equiv \hbar^2 (\bar{\alpha}^D P_{DD'} \alpha^{D'})^{-2} L_a [(2J+1)(2J+2)]^{-1} (J+1-\alpha)(J-j_{\max}), \quad (2.67)$$

$$\bar{K}^a \equiv \hbar^2 [(2J+2)(2J+1)]^{-1} (J+1+\alpha)(J+j_{\max}+2)(J+1-j_{\min})(J+1+j_{\min}) \tilde{L}^a \quad (2.68)$$

acting on states $|j, q\rangle$ normalized according to

$$\langle j, q | j, q\rangle = e[(2j+1)(2j-q)!(j_{\max}+\alpha)!(\alpha+j_{\max}+1)!]^{-1} (\bar{\alpha}^D P_{DD'} \alpha^{D'})^{2j} q! \times (j+j_{\min})!(j-j_{\min})!(j_{\max}-j)!(j+j_{\max}+1)!(j+\alpha)!(\alpha-j-1)!, \quad (2.69)$$

e being a real positive constant. K^a raises spin by one unit, $\bar{K}^a$ lowers spin by one unit. Also

$$C_2 = \tfrac{1}{2}\hbar^2 [j_{\min}^2 + j_{\max}(j_{\max}+2) + \alpha^2], \quad (2.70)$$

$$C_3^2 = \hbar^6 \alpha^2 (j_{\max}+1)^2 j_{\min}^2, \quad (2.71)$$

$$C_4 = C_2^2 + 2\hbar^2 C_2 - \hbar^4 [j_{\min}^2 (j_{\max}+1)^2 + (j_{\max}+1)^2 \alpha^2 + \alpha^2 j_{\min}^2]. \quad (2.72)$$

These formulae are valid even when only one spin is present in the representation (of course K^a and $\bar{K}^a$ then vanish identically).

3. Spinor intertwining operators for $\mathscr{R}$-algebra representations

The discussion begins with the following simple algebraic lemma.

Lemma. Let $\frac{1}{4}k$, $\frac{1}{4}l$, $\frac{1}{4}m$, $\frac{1}{4}p$ be the roots of the polynomial equation

$$x^4 + \rho x^2 + \sigma x + \tau = 0 \quad (3.1)$$

with ρ, σ, and τ complex. Note that $k+l+m+p = 0$.

Define a, b and c by the equations

$$a \equiv \tfrac{1}{4}(l+p) = -\tfrac{1}{4}(m+k), \quad (3.2)$$

$$b \equiv \tfrac{1}{4}(l+m) = -\tfrac{1}{4}(k+p), \tag{3.3}$$

$$c \equiv \tfrac{1}{4}(m+p) = -\tfrac{1}{4}(k+l). \tag{3.4}$$

Then $\pm a$, $\pm b$, $\pm c$ are the roots of the polynomial equation

$$x^6 + 2\rho x^4 + (\rho^2 - 4\tau)\, x^2 - \sigma^2 = 0. \tag{3.5}$$

Conversely let $\pm a$, $\pm b$, $\pm c$ be the roots of (3.5) with the sign of σ fixed by $\sigma = abc$. Then

$$\tfrac{1}{4}k = \tfrac{1}{2}(\pm a \pm b \pm c), \tag{3.6}$$

$$-\tfrac{1}{4}l = \tfrac{1}{2}(\pm a \pm b \mp c), \tag{3.7}$$

$$-\tfrac{1}{4}m = \tfrac{1}{2}(\mp a \pm b \pm c), \tag{3.8}$$

$$-\tfrac{1}{4}p = \tfrac{1}{2}(\pm a \mp b \pm c), \tag{3.9}$$

are the roots of (3.1), where the $\pm$ signs are independent horizontally, but related vertically.

The six permutations of a, b, and c together with the eight random sign changes $\pm a$, $\pm b$, $\pm c$ give forty-eight different transformations altogether.

These agree one to one with the twenty-four permutations of k, l, m, and p, together with the simultaneous sign change $k, l, m, p \to -k, -l, -m, -p$.

The application here is to the polynomial

$$\phi(x) \equiv x^6 - x^4(2c_2+1) - x^2(c_4 - c_2^2 - 2c_2) - c_3^2 \tag{3.10}$$

(so $\phi(x) \equiv f(j)$ in the notation of § 2). One may choose the roots $\pm a$, $\pm b$, $\pm c$, of $\phi(x) = 0$, such that $abc = c_3$. Then $\tfrac{1}{4}k$, $\tfrac{1}{4}l$, $\tfrac{1}{4}m$, $\tfrac{1}{4}p$ are the roots of

$$x^4 - \tfrac{1}{2}(2c_2+1)\, x^2 + c_3 x - \tfrac{1}{4}(c_4 - c_2 + \tfrac{1}{4}) = 0. \tag{3.11}$$

Consider from now onwards the bounded spin-representations of the $\mathscr{R}$-algebra. Then the roots of $\phi(x) = 0$ are $x = \pm j_{\min}$, $= \pm(j_{\max}+1)$ and $\pm\alpha$, with $\alpha > j_{\max}$. (For convenience, one excludes those representations containing only one spin, which do not satisfy these criteria, so excluding $\alpha_0 \leqslant 0$ and $2\beta_{j_0} \leqslant j_0$ in the notation of § 2.)

To take account of the sign of c_3, it is often convenient to replace $j_{\min}$, $j_{\max}$ and α by closely related quantities r, s, and d defined by

$$j_{\min} = \tfrac{1}{2}|s-r|, \tag{3.12}$$

$$j_{\max} = \tfrac{1}{2}(s+r), \tag{3.13}$$

$$\alpha = \tfrac{1}{2}(s+r) + 2d + 2, \tag{3.14}$$

$$c_3 = \tfrac{1}{8}(s-r)\,(s+r+2)\,(s+r+2d+4). \tag{3.15}$$

Note that $d > -1$ and that (3.15) serves only to fix the sign of $s-r$; r and s are non-negative integers.

Then k, l, m and p are, in some order, $3r+s+4d+6$, $-(3s+r+4d+6)$, $-(r-s+4d+2)$ and $s-r+4d+2$. Since k, l, m and p define r, s and d and they in turn define $j_{\min}$, $j_{\max}$, α and c_3, then conversely, one may use k, l, m and p together or r, s and d together to characterize the $\mathscr{R}$-algebra representation under consideration. For definiteness, put

$$k = 3r+s+4d+6, \tag{3.16}$$

$$l = -(3s+r+4d+6), \tag{3.17}$$

$$m = -(r-s+4d+2), \tag{3.18}$$

$$p = s-r+4d+2. \tag{3.19}$$

Note that
$$\tfrac{1}{4}(m+p) = -\tfrac{1}{4}(k+l) = \tfrac{1}{2}(s-r) = \pm j_{\min}, \tag{3.20}$$

$$\tfrac{1}{4}(l+p) = -\tfrac{1}{4}(k+m) = -\tfrac{1}{2}(s+r+2) = -j_{\max}-1, \tag{3.21}$$

$$\tfrac{1}{4}(k+p) = -\tfrac{1}{4}(l+m) = \tfrac{1}{2}(s+r)+(2d+2) = \alpha, \tag{3.22}$$

$$r+1 = \tfrac{1}{4}(2k+l+m), \tag{3.23}$$

$$s+1 = \tfrac{1}{4}(m-l), \tag{3.24}$$

$$2d+1 = \tfrac{1}{4}(2p+k+l). \tag{3.25}$$

States in a given representation may be labelled $|c_2, c_3, c_4, j, q\rangle$, $|d, r, s, j, q\rangle$ or equivalently $|k, l, m, p, j, q\rangle$.

The aim now is to clarify the structure of the R^a operator in an $\mathscr{R}$-algebra representation by factorizing it by means of spinor intertwining operators. By using the $|d, r, s, j, q\rangle$ notation, define, in succession,

$$l^A\,|d,r,s,j,q\rangle = (\bar{\alpha}^D P_{DD'}\alpha^{D'})^{-1}[\alpha^{A'}P_{A'}{}^A\,|d-\tfrac{1}{2},r+1,s,j+\tfrac{1}{2},q+1\rangle + (2j+1-q)\,\bar{\alpha}^A\,|d-\tfrac{1}{2},r+1,s,j+\tfrac{1}{2},q\rangle], \tag{3.26}$$

$$m^A\,|d,r,s,j,q\rangle = P^A_{A'}\alpha^{A'}\,|d-\tfrac{1}{2},r+1,s,j-\tfrac{1}{2},q\rangle - q\bar{\alpha}^A\,|d-\tfrac{1}{2},r+1,s,j-\tfrac{1}{2},q-1\rangle, \tag{3.27}$$

$$l^{A'}\,|d,r,s,j,q\rangle = \alpha^{A'}\,|d+\tfrac{1}{2},r-1,s,j-\tfrac{1}{2},q\rangle + qP^{A'}{}_A\bar{\alpha}^A\,|d+\tfrac{1}{2},r-1,s,j-\tfrac{1}{2},q-1\rangle, \tag{3.28}$$

$$m^{A'}\,|d,r,s,j,q\rangle = (\bar{\alpha}^D P_{DD'}\alpha^{D'})^{-1}[\alpha^{A'}\,|d+\tfrac{1}{2},r-1,s,j+\tfrac{1}{2},q+1\rangle + (2j+1-q)\,P^{A'}{}_A\alpha^A\,|d+\tfrac{1}{2},r-1,s,j+\tfrac{1}{2},q\rangle], \tag{3.29}$$

with scalar product

$$\langle d,r,s,j,q|d,r,s,j,q\rangle = e(d,r,s,j)\;q!\,(\bar{\alpha}^D P_{DD'}\alpha^{D'})^{2j}\,[(2j+1)\,(2j-q)!]^{-1}. \tag{3.30}$$

In terms of the k, l, m, p notation, these read

$$l^A\,|k,l,m,p,j,q\rangle = (\bar{\alpha}^D P_{DD'}\alpha^{D'})^{-1}[\alpha^{B'}P_{B'}{}^A\,|k+1,l+1,m+1,p-3,j+\tfrac{1}{2},q+1\rangle + (2j+1-q)\,\bar{\alpha}^A\,|k+1,l+1,m+1,p-3,\;\;j+\tfrac{1}{2},q\rangle], \tag{3.31}$$

$$m^A\,|k,l,m,p,j,q\rangle = P^A_{A'}\alpha^{A'}\,|k+1,l+1,m+1,p-3,j-\tfrac{1}{2},q\rangle - q\bar{\alpha}^A\,|k+1,l+1,m+1,p-3,j-\tfrac{1}{2},q-1\rangle, \tag{3.32}$$

$$l^{A'}\,|k,l,m,p,j,q\rangle = \alpha^{A'}\,|k-1,l-1,m-1,p+3,j-\tfrac{1}{2},q\rangle + qP^{A'}_A\bar{\alpha}^A\,|k-1,l-1,m-1,p+3,j-\tfrac{1}{2},q-1\rangle, \tag{3.33}$$

$$m^{A'}\,|k,l,m,p,j,q\rangle = (\bar{\alpha}^D P_{DD'}\alpha^{D'})^{-1}[-\alpha^{A'}|k-1,l-1,m-1,p+3,j+\tfrac{1}{2},q+1\rangle + (2j+1-q)\,P^{A'}_A\,\bar{\alpha}^A\,|k-1,l-1,m-1,p+3,j+\tfrac{1}{2},q], \tag{3.34}$$

with scalar product

$$\langle k,l,m,p,j,q|k,l,m,p,j,q\rangle \equiv e(k,l,m,p,j)\,q!(\bar{\alpha}^D P_{DD'}\alpha^{D'})^{2j}\,[(2j+1)\,(2j-q)!]^{-1}. \tag{3.35}$$

Note that l^A, m^A, $l^{A'}$ and $m^{A'}$ are intertwining operators, since they change the d, r, s or k, l, m, p values.

Although one might perhaps prefer the d, r, s representation, since at least r and s have a simple interpretation, the remaining formulae will be given only in the k, l, m, p representation, for reasons which will eventually become clear. Of course, any such formula may be translated immediately into the d, r, s representation.

The intertwining operators $l^{A'}$ and $m^{A'}$ are not the adjoints of l^A and m^A respectively. In fact, using (3.31)–(3.35) one finds easily that

$$\bar{l}^{A'} = l^{A'}e(K,L,M,P,J)\,[e(K-1,L-1,M-1,P+3,J-\tfrac{1}{2})]^{-1}\,2J(2J+1)^{-1}$$
$$= e(K+1,L+1,M+1,P-3,J+\tfrac{1}{2})\,[e(K,L,M,P,J)]^{-1}\,(2J+1)\,(2J+2)^{-1}\,l^{A'}, \quad (3.36)$$

$$\bar{l}^{A} = l^{A}e(K,L,M,P,J)\,[e(K+1,L+1,M+1,P-3,J+\tfrac{1}{2})]^{-1}\,(2J+2)\,(2J+1)^{-1}$$
$$= e(K-1,L-1,M-1,P+3,J-\tfrac{1}{2})\,[e(K,L,M,P,J)]^{-1}\,(2J+1)\,(2J)^{-1}\,l^{A}, \quad (3.37)$$

$$\bar{m}^{A'} = m^{A'}e(K,L,M,P,J)\,[e(K-1,L-1,M-1,P+3,J+\tfrac{1}{2})]^{-1}\,(2J+2)\,(2J+1)^{-1}$$
$$= e(K+1,L+1,M+1,P-3,J-\tfrac{1}{2})\,[e(K,L,M,P,J)]^{-1}\,(2J+1)\,(2J)^{-1}\,m^{A'}, \quad (3.38)$$

$$\bar{m}^{A} = m^{A}e(K,L,M,P,J)\,[e(K+1,L+1,M+1,P-3,J-\tfrac{1}{2})]^{-1}(2J)\,(2J+1)^{-1}$$
$$= e(K-1,L-1,M-1,P+3,J+\tfrac{1}{2})\,[e(K,L,M,P,J)]^{-1}\,(2J+1)\,(2J+2)^{-1}\,m^{A}. \quad (3.39)$$

The basic equation defining the $\mathscr{R}$-algebra representation is, from (2.66)–(2.68),

$$R^a = \hbar^2\alpha(K,L,M,P,J)\,P^a + \hbar\beta(K,L,M,P,J)\,S^a$$
$$+\hbar^2 L^a(\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^{-2}\,\gamma(K,L,M,P,J)\,[(2J+1)\,(2J+2)]^{-1}$$
$$+\hbar^2\tilde{L}^a\delta(K,L,M,P,J)\,(2J+1)^{-1}, \quad (3.40)$$

where from (2.28), (2.29), (2.45), (2.64)–(2.65) and (3.20)–(3.22),

$$\alpha(K,L,M,P,J) = \tfrac{1}{2}\hbar^{-2}C_2 - \tfrac{1}{2}J^2 - \tfrac{1}{2}J$$
$$= -\tfrac{1}{32}(KL+LM+MP+PK+KM+LP) - \tfrac{1}{2}J^2 - \tfrac{1}{2}J - \tfrac{1}{4}, \quad (3.41)$$

$$2J(J+1)\,\beta(K,L,M,P,J) = -\hbar^{-3}C_3 = \tfrac{1}{64}(KLM+KLP+KMP+LMP), \quad (3.42)$$

$$2(J+1)\,\delta(K,L,M,P,J+1)\,\gamma(K,L,M,P,J)$$
$$= [J+1-\tfrac{1}{4}(L+M)]\,[J+1-\tfrac{1}{4}(K+L)]\,[J+1-\tfrac{1}{4}(M+P)]$$
$$\times[J+1-\tfrac{1}{4}(K+M)]\,[J+1-\tfrac{1}{4}(L+P)]\,[J+1-\tfrac{1}{4}(K+P)]. \quad (3.43)$$

Here, K, L, M, P, J, $e(K, L, M, P, J)$, $\alpha(K, L, M, P, J)$, etc., are the operators whose eigenvalues on $|k, l, m, p, j, q\rangle$ are respectively k, l, m, p, j, $e(k, l, m, p, j)$, $\alpha(k, l, m, p, j)$, etc.

l^A, m^A, $l^{A'}$, $m^{A'}$ are generic spin-raising operators. The aim is to multiply them by suitable functions of k, l, m, p, and j, and combine them suitably to achieve the factorization of the R^a operator. The key to success is to notice that L^a and $\tilde{L}^a$, as defined in (2.9) and (2.10), are easily factorized: one checks that

$$L^a(\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^{-2} = l^A m^{A'}, \quad (3.44)$$

$$\tilde{L}^a = m^A l^{A'}. \quad (3.45)$$

In addition one sees that

$$l^A l^{A'} = \hbar^{-1}S^a + JP^a, \quad l^{A'} l^A = \hbar^{-1}S^a + (J+1)\,P^a, \quad (3.46)$$

$$m^A m^{A'} = -\hbar^{-1}S^a + (J+1)\,P^a, \quad m^{A'} m^A = -\hbar^{-1}S^a + JP^a, \quad (3.47)$$

$$[l^A, l^B] = 0 = [m^A, m^B], \quad [l^{A'}, l^{B'}] = 0 = [m^{A'}, m^{B'}], \quad (3.48)$$

$$[m^A, l^{A'}] = 0 = [l^A, m^{A'}], \quad (3.49)$$

$$l^A m^B|k,l,m,p,j,q\rangle = (\hbar^{-1}S^{AA'}P_{A'}{}^B - J\epsilon^{AB})|k+2,l+2,m+2,p-6,j,q\rangle, \tag{3.50}$$

$$m^A l^B|k,l,m,p,j,q\rangle = (\hbar^{-1}S^{AA'}P_{A'}{}^B + (J+1)\,\epsilon^{AB})|k+2,l+2,m+2,p-6,j,q\rangle, \tag{3.51}$$

$$l^{A'} m^{B'}|k,l,m,p,j,q\rangle = (\hbar^{-1}S^{AA'}P_A{}^{B'} - (J+1)\,\epsilon^{A'B'})|k-2,l-2,m-2,p+6,j,q\rangle, \tag{3.52}$$

$$m^{A'} l^{B'}|k,l,m,p,j,q\rangle = (\hbar^{-1}S^{A'A}P_A{}^{B'} + J\epsilon^{A'B'})|k-2,l-2,m-2,p+6,j,q\rangle, \tag{3.53}$$

$$l_A S^{AA'} = -\hbar l_A P^{AA'} J, \qquad [l^A, S^{BB'}] = -\hbar l^B P^{AB'} + \tfrac{1}{2}\hbar l^A P^{BB'}, \tag{3.54}$$

$$m^A S^{AA'} = \hbar m_A P^{AA'}(J+1), \qquad [m^A, S^{BB'}] = -\hbar m^B P^{AB'} + \tfrac{1}{2}\hbar m^A P^{BB'}, \tag{3.55}$$

$$l_{A'} S^{AA'} = -\hbar l_A P^{AA'}(J+1), \qquad [l^{A'}, S^{BB'}] = \hbar l^{B'} P^{AB'} - \tfrac{1}{2}\hbar l^{A'} P^{BB'}, \tag{3.56}$$

$$m_{A'} S^{AA'} = \hbar m_{A'} P^{AA'} J, \qquad [m^{A'}, S^{BB'}] = \hbar m^{B'} P^{AB'} - \tfrac{1}{2}\hbar m^{A'} P^{BB'}. \tag{3.57}$$

Next define

$$\tilde{l}^A \equiv \hbar l^A a(K,L,M,P,J)\,(2J+1)^{-1} = \hbar a(K-1,L-1,M-1,P+3,J-\tfrac{1}{2})\,(2J)^{-1}\,l^A, \tag{3.58}$$

$$\tilde{m}^A \equiv \hbar m^A c(K,L,M,P,J)\,(2J+1)^{-1} = \hbar c(K-1,L-1,M-1,P+3,J-\tfrac{1}{2})\,(2J+2)^{-1}\,m^A, \tag{3.59}$$

$$\tilde{l}^{A'} \equiv \hbar l^{A'} b(K,L,M,P,J)\,(2J+1)^{-1} = \hbar b(K+1,L+1,M+1,P-3,J+\tfrac{1}{2})\,(2J+2)^{-1}\,l^{A'}, \tag{3.60}$$

$$\tilde{m}^{A'} \equiv \hbar m^{A'} d(K,L,M,P,J)\,(2J+1)^{-1} = \hbar d(K+1,L+1,M+1,P-3,J+\tfrac{1}{2})\,(2J)^{-1}\,m^{A'}, \tag{3.61}$$

where $a(K, L, M, P, J)$, $b(K, L, M, P, J)$, $c(K, L, M, P, J)$ and $d(K, L, M, P, J)$ are suitable Clebsch–Gordan coefficients to be found.

First require that

$$\begin{aligned}\tilde{l}^A\tilde{m}^{A'} = K^a &= \hbar^2(\bar{\alpha}^D P_{DD'}\alpha^{D'})^{-2} L^a\,\gamma(K,L,M,P,J)\,[(2J+1)\,(2J+2)]^{-1}\\ &= \hbar^2(\bar{\alpha}^D P_{DD'}\alpha^{D'})^{-2}\,\gamma(K,L,M,P,J-1)\,[(2J-1)\,(2J)]^{-1}\,L^a,\end{aligned} \tag{3.62}$$

$$\begin{aligned}\tilde{m}^A\tilde{l}^{A'} = K^a &= \hbar^2\tilde{L}^a\,\delta(K,L,M,P,J)\,(2J+1)^{-1}\\ &= \hbar^2\delta(K,L,M,P,J+1)\,(2J+3)^{-1}\,\tilde{L}^a.\end{aligned} \tag{3.63}$$

By using (3.44) and (3.45), this forces

$$a(K-1,L-1,M-1,P+3,J+\tfrac{1}{2})\,d(K,L,M,P,J) = \gamma(K,L,M,P,J), \tag{3.64}$$

$$c(K-1,L-1,M-1,P+3,J-\tfrac{1}{2})\,b(K,L,M,P,J) = 2J\delta(K,L,M,P,J). \tag{3.65}$$

Next define

$$r^A \equiv \tilde{l}^A + \tilde{m}^A, \quad r^{A'} \equiv \tilde{l}^{A'} + \tilde{m}^{A'}. \tag{3.66}$$

Then using (3.40), (3.46) and (3.47) one finds that

$$\begin{aligned} r^A r^{A'} = R^a &- \hbar^2\alpha(K,L,M,P,J)\,P^a - \hbar\beta(K,L,M,P,J)\,S^a\\ &+ \frac{\hbar^2}{4(2J+1)}\left(\frac{\hbar S^a}{\vec{J}^2} + 2P^a\right)[a(K-1,\,L-1,\,M-1,\,P+3,\,J-\tfrac{1}{2})\,b(K,\,L,\,M,\,P,\,J)\\ &+ c(K-1,\,L-1,\,M-1,\,P+3,\,J+\tfrac{1}{2})\,d(K,\,L,\,M,\,P,\,J)]\\ &+ \frac{\hbar^3}{4\vec{J}^2}S^a[a(K-1,\,L-1,\,M-1,\,P+3,\,J-\tfrac{1}{2})\,b(K,\,L,\,M,\,P,\,J)\\ &- c(K-1,\,L-1,\,M-1,\,P+3,\,J+\tfrac{1}{2})\,d(K,\,L,\,M,\,P,\,J)].\end{aligned} \tag{3.67}$$

Next require that $r^{A'}$ be the adjoint of r^A, $r^{A'} = \bar{r}^{A'}$. This forces $\overline{\tilde{l}^{A'}} = \tilde{l}^A$, $\overline{\tilde{m}^{A'}} = \tilde{m}^A$. Now

$$\begin{aligned}\overline{\tilde{l}^{A'}} = \bar{b}(K,L,M,P,J)\,&[a(K-1,L-1,M-1,P+3,J-\tfrac{1}{2})]^{-1}\\ &\times e(K-1,L-1,M-1,P+3,J-\tfrac{1}{2})\,[e(K,L,M,P,J)]^{-1}\,\tilde{l}^A,\end{aligned} \tag{3.68}$$

$$\overline{\tilde{m}^{A'}} = \bar{d}(K,L,M,P,J)\,[c(K-1,L-1,M-1,P+3,J+\tfrac{1}{2})]^{-1}$$
$$\times\, e(K-1,L-1,M-1,P+3,J+\tfrac{1}{2})\,[e(K,L,M,P,J)]^{-1}\,\tilde{m}^{A}. \tag{3.69}$$

Hence $\bar{r}^{A'} = r^{A'}$ if and only if

$$a(K-1,L-1,M-1,P+3,J-\tfrac{1}{2})\,[e(K-1,L-1,M-1,P+3,J-\tfrac{1}{2})]^{-1}$$
$$= \bar{b}(K,L,M,P,J)\,[e(K,L,M,P,J)]^{-1}, \tag{3.70}$$

$$c(K-1,L-1,M-1,P+3,J+\tfrac{1}{2})\,[e(K-1,L-1,M-1,P+3,J+\tfrac{1}{2})]^{-1}$$
$$= \bar{d}(K,L,M,P,J)\,[e(K,L,M,P,J)]^{-1}. \tag{3.71}$$

Then (3.64), (3.65), (3.70) and (3.71) give

$$(2J+2)\,\delta(K,L,M,P,J+1)[\bar{\gamma}(K,L,M,P,J)]^{-1} = e(K,L,M,P,J+1)\,[e(K,L,M,P,J)]^{-1}, \tag{3.72}$$

in agreement with (2.51).

Let $\lambda(J)$ and $\mu(J)$ be given by

$$\lambda(J) \equiv a(K-1,L-1,M-1,P+3,J-\tfrac{1}{2})\; b(K,L,M,P,J), \tag{3.73}$$

$$\mu(J) \equiv c(K-1,L-1,M-1,P+3,J+\tfrac{1}{2})\, d(K,L,M,P,J). \tag{3.74}$$

Then (3.64) and (3.65) give

$$\lambda(J+1)\,\mu(J) = (2J+2)\,\delta(K,L,M,P,J+1)\;\gamma(K,L,M,P,J). \tag{3.75}$$

Also the expression for $r^{A}r^{A'}$, (3.67) reads

$$r^{A}r^{A'} = R^{a} - \hbar^{2}\alpha(K,L,M,P,J)\,P^{a} - \hbar\beta(K,L,M,P,J)\,S^{a} + \tfrac{1}{4}\hbar^{3}S^{a}\vec{J}^{-2}[\lambda(J)-\mu(J)]$$
$$+\tfrac{1}{4}\hbar^{2}(2J+1)^{-1}\,(\hbar S^{a}\vec{J}^{-2}+2P^{a})\,[\lambda(J)+\mu(J)]. \tag{3.76}$$

In (3.76) one can eliminate the pole at $2J+1 = 0$ by requiring

$$\lambda(-\tfrac{1}{2}) + \mu(-\tfrac{1}{2}) = 0. \tag{3.77}$$

Take
$$\lambda(J) = -[J-\tfrac{1}{4}(K+P)]\,[J-\tfrac{1}{4}(M+P)]\,[J-\tfrac{1}{4}(L+P)], \tag{3.78}$$

$$\mu(J) = -[J+1+\tfrac{1}{4}(M+P)]\,[J+1+\tfrac{1}{4}(K+P)]\,[J+1+\tfrac{1}{4}(L+P)]. \tag{3.79}$$

Then looking at (3.43) one sees that (3.75) is satisfied, as is (3.77). From (3.78) and (3.79), $\lambda(J) = -\mu(-J-1)$, so $\lambda(J)+\mu(J)$ and $\lambda(J)-\mu(J)$ are respectively even and odd functions of $2J+1$:

$$\lambda(J)+\mu(J) = (2J+1)\,[-J(J+1)+\lambda(0)-\mu(0)]$$
$$= (2J+1)\,[-J(J+1)+\hbar^{-2}C_{2}-\tfrac{1}{8}(P+2)^{2}], \tag{3.80}$$

$$\lambda(J)-\mu(J) = J(J+1)\,(P+3)+\lambda(0)-\mu(0)$$
$$= (P+3)\,J(J+1)-\hbar^{-2}C_{2}+\tfrac{1}{8}(P+2)^{2}-2\hbar^{-3}C_{3}. \tag{3.81}$$

Substituting (3.80) and (3.81) into (3.76) and using (3.41) and (3.42) one finds

$$R^{a} = r^{A}r^{A'} + \tfrac{1}{16}\hbar^{2}P^{a}(P+2)^{2} - \tfrac{1}{4}\hbar S^{a}(P+2). \tag{3.82}$$

This is the fundamental equation on which the whole of the present theory hangs. One can argue that the choice of $\lambda(J)$ and $\mu(J)$ was not unique – they were essentially chosen from thin air. There might be other ways of factorizing R^{a} with different remainders. However, given equation (3.82), by looking at (3.76) it is easy to see that $\lambda(J)$ and $\mu(J)$ are fixed. Thus the question is: what is special about the particular choice of coefficients in (3.82)? The only answer that is available at the moment (apart from the elegance of the subsequent theory!), is that this is the equation that the three-twistor model of particles provides. The equation itself, however, does not depend on any model. One can see also that any other choice of $\lambda(J)$ and $\mu(J)$ would give a more complicated expression for the remainder term.

Now, taking (3.82) as the correct equation, one sees that, working backwards, $\lambda(J)$ and $\mu(J)$ are determined uniquely, but a, b, c, and d are not. However, the remaining freedom in choice of the functions a, b, c, and d is easily seen to be equivalent to the freedom to rescale the states in the different representations. Thus, given (3.82) and given that r^A raises k, l, and m by one unit, while lowering p by three units, the solution is essentially unique.

Accordingly, to solve all equations one may take

$$a(K,L,M,P,J) = -[J+1-\tfrac{1}{4}(K+P)], \tag{3.83}$$

$$b(K,L,M,P,J) = [J-\tfrac{1}{4}(M+P)]\,[J-\tfrac{1}{4}(L+P)], \tag{3.84}$$

$$c(K,L,M,P,J) = [J+\tfrac{1}{4}(M+P)]\,[J+\tfrac{1}{4}(K+P)], \tag{3.85}$$

$$d(K,L,M,P,J) = -[J+1+\tfrac{1}{4}(L+P)]. \tag{3.86}$$

Then from (3.64) and (3.65)

$$\gamma(K,L,M,P,J) = [J+1-\tfrac{1}{4}(K+P)]\,[J+1+\tfrac{1}{4}(L+P)], \tag{3.87}$$

$$\begin{aligned} 2J\delta(K,L,M,P,J) &= [J+\tfrac{1}{4}(M+P)]\,[J+\tfrac{1}{4}(K+P)] \\ &\quad +[J-\tfrac{1}{4}(M+P)]\,[J-\tfrac{1}{4}(L+P)], \end{aligned} \tag{3.88}$$

$$\begin{aligned} e(K,L,M,P,J) &= [(K-L)!]^{-2}g(K-L,L-M)\,[\tfrac{1}{4}(K+P)-J-1]! \\ &\quad \times[J+\tfrac{1}{4}(K+P)]!\,[J-\tfrac{1}{4}(L+P)]!\,[-\tfrac{1}{4}(L+P)-J-1]! \\ &\quad \times[J+\tfrac{1}{4}(M+P)]!\,[J-\tfrac{1}{4}(M+P)]!, \end{aligned} \tag{3.89}$$

where $g(K-L,L-M)$ is an arbitrary positive function. Equations (3.87), (3.88), and (3.89) are in complete agreement with equations (2.64), (2.65) and (2.69), with the translation given by (3.20), (3.21) and (3.22).

Next, by using the properties of l^A, m^A, $l^{A'}$, and $m^{A'}$, listed in (3.46)–(3.53) it is merely a matter of algebra to compute the commutation properties of the operators r^A and $r^{A'}$. One finds that

$$[r^A, r^B] = 0, \quad [r^{A'}, r^{B'}] = 0, \tag{3.90}$$

$$[r^A, r^{A'}] = \hbar S^a - \tfrac{1}{2}\hbar^2 P P^a, \tag{3.91}$$

$$[r^A, P] = 3r^A, \quad [r^{A'}, P] = -3r^{A'}, \tag{3.92}$$

$$[r^A, K] = [r^A, L] = [r^A, M] = -r^A, \tag{3.93}$$

$$[r^{A'}, K] = [r^{A'}, L] = [r^{A'}, M] = r^{A'}, \tag{3.94}$$

together with the Poincaré relations

$$[P^b, r^A] = 0, \quad [P^b, r^{A'}] = 0, \tag{3.95}$$

$$[S^b, r^A] = \hbar r^B P^{AB'} - \tfrac{1}{2}\hbar r^A P^{BB'}, \tag{3.96}$$

$$[S^b, r^{A'}] = -\hbar r^{B'} P^{A'B} + \tfrac{1}{2}\hbar r^{A'} P^{BB'}, \tag{3.97}$$

$$[S_a, S_b] = \hbar\epsilon_{AB} S_{C(A'} P^C{}_{B')} - \hbar\epsilon_{A'B'} S_{C'(A} P_{B)}{}^{C'}. \tag{3.98}$$

Now from the point of view of the little algebra, $P^{AA'}$ may be regarded as a system of four numbers, rather than as an operator.

Then the above relations (3.90), (3.91), (3.92), (3.96), (3.97) and (3.98) show that r^A, $r^{A'}$, $S^{AA'}$ and P form a Lie algebra under commutation. Alternatively if one wishes to eliminate P^a, one may use P^a to project primed spinor indices into the unprimed spin space: define

$$S_A{}^B \equiv S_{AA'} P^{BA'} = S^B{}_A, \tag{3.99}$$

$$\bar{r}^A \equiv -r^{A'} P_{A'}{}^A. \tag{3.100}$$

Then (3.90), (3.91), (3.92), (3.96), (3.97) and (3.98) become

$$[S_A{}^B, S_C{}^D] = \hbar(\delta_C{}^B S_A{}^D - \delta_A{}^D S_C{}^B), \tag{3.101}$$

$$[S_C{}^B, r_A] = \hbar(\delta_A{}^B r_C - \tfrac{1}{2} r_A \delta_C{}^B), \tag{3.102}$$

$$[S_C{}^B, \bar{r}_A] = \hbar(\delta_A{}^B \bar{r}_C - \tfrac{1}{2}\bar{r}_A \delta_C{}^B), \tag{3.103}$$

$$[r_A, \bar{r}^B] = \hbar S_A{}^B - \tfrac{1}{2}\hbar^2 P \delta_A{}^B, \tag{3.104}$$

$$[r_A, \tfrac{1}{3}P] = r_A, \quad [\bar{r}_A, \tfrac{1}{3}P] = -\bar{r}_A. \tag{3.105}$$

Also define

$$T_B{}^A \equiv S_B{}^A - \tfrac{1}{6}\delta_B{}^A P\hbar = S_B{}^A + \tfrac{1}{2} T \delta_B{}^A, \tag{3.106}$$

$$T \equiv T_A{}^A \equiv -\tfrac{1}{3}P\hbar. \tag{3.107}$$

Then

$$[r^A, T] = -r_A \hbar, \quad [\bar{r}_A, T] = \bar{r}_A \hbar, \tag{3.108}$$

$$[T_A{}^B, T_C{}^D] = \hbar(\delta_C{}^B T_A{}^D - \delta_A{}^D T_C{}^B), \tag{3.109}$$

$$[r_A, T_C{}^B] = -\hbar\delta_A{}^B \bar{r}_C, \tag{3.110}$$

$$[\bar{r}^B, T_C{}^A] = \hbar\delta_C{}^B \bar{r}^A, \tag{3.111}$$

$$[r_A, \bar{r}^B] = \hbar T_A{}^B + \hbar\delta_A{}^B T. \tag{3.112}$$

Now define the (3×3) matrix operator

$$A^i_j \equiv \begin{pmatrix} -T & r_A \\ \bar{r}^B & T_A{}^B \end{pmatrix}. \tag{3.113}$$

Then A^i_j is trace-free and Hermitian, and equations (3.108)–(3.112) become just

$$[A^i_j, A^k_l] = \hbar(\delta^i_l A^k_j - \delta^k_j A^i_l). \tag{3.114}$$

Hence A^i_j generates the SU(3) Lie algebra! The SU(3) structure is naturally broken by the formalism (graded) down to SU(2) × U(1), T generating the U(1) piece and $S_A{}^B$ generating the 'isospin' subgroup. Thus the relativistic spin vector is, for this SU(3), precisely the 'isospin' operator!

SU(3) has two Casimir operators. The next task is to relate these to the $\mathscr{R}$-algebra Casimir operators. One has

$$S_A{}^B S_B{}^A = 2\vec{J}^2, \tag{3.115}$$

$$r_A \bar{r}^A = C_2 - \vec{J}^2 - \tfrac{1}{8}\hbar^2(P+2)^2, \tag{3.116}$$

$$r_A \bar{r}_B S^{AB} = C_3 - \tfrac{1}{2}\hbar(P+2)\vec{J}^2, \tag{3.117}$$

$$-\bar{r}^B T r_B = -(T+\hbar)\, r_A \bar{r}^A + 3\hbar T(T+\hbar), \tag{3.118}$$

$$-r_A \bar{r}^A T = -T r_A \bar{r}^A, \tag{3.119}$$

$$r_A T_B{}^A \bar{r}^B = r_A \bar{r}^B T_B{}^A - 2\hbar r_A \bar{r}^A, \tag{3.120}$$

$$\bar{r}^B r_A T_B{}^A = T_B{}^A \bar{r}^A r_B = -r_A \bar{r}_B S^{AB} + \tfrac{1}{2} T r_A \bar{r}^A - \tfrac{3}{2}\hbar T^2 - \hbar S_A{}^B S_B{}^A, \tag{3.121}$$

$$T_A{}^B T_C{}^A T_B{}^C = \tfrac{1}{4}T^3 + S_B{}^A S_A{}^B(\tfrac{3}{2}T - \hbar), \tag{3.122}$$

$$T_A{}^B T_B{}^A = S_A{}^B S_B{}^A + \tfrac{1}{2}T^2, \tag{3.123}$$

$$\bar{r}^B r_B = r_B \bar{r}^B - 3\hbar T. \tag{3.124}$$

Then the second- and third-order SU(3) Casimir operators, D_2 and D_3 are given by, by using (3.113)–(3.124),

$$\begin{aligned}\hbar^2 D_2 &\equiv A^i_j A^j_i \\ &= T^2 + r_A \bar{r}^A + \bar{r}^B r_B + T_A{}^B T_B{}^A \\ &= 2C_2 - \hbar^2 - \tfrac{1}{12}\hbar^2 P^2,\end{aligned} \tag{3.125}$$

$$\begin{aligned}\hbar^3 D_3 &\equiv A^i_j A^j_k A^k_i + \tfrac{3}{2}\hbar A^i_j A^j_i \\ &= -T^3 - T r_A \bar{r}^A - \bar{r}^A T r_A - r_A \bar{r}^A T + T_A{}^B \bar{r}^A r_B \\ &\quad + r_B T_A{}^B \bar{r}^A + \bar{r}^A r_B T_A{}^B + T_B{}^A T_C{}^B T_A{}^C + \tfrac{3}{2}\hbar^3 D_2 \\ &= -3C_3 - \tfrac{3}{2} T C_2 + \tfrac{15}{16} T^3 - \tfrac{3}{4}\hbar^2 T \\ &= -3C_3 + \tfrac{1}{2}P\hbar C_2 - \tfrac{5}{144}\hbar^3 P + \tfrac{1}{4}\hbar^3 P.\end{aligned} \tag{3.126}$$

Standard SU(3) representation theory characterizes the unitary representations in terms of two non-negative integers λ and μ obeying

$$\begin{aligned}D_2 &= \tfrac{2}{3}(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) \\ &= -2 - \tfrac{2}{9}[-(\mu-\lambda)(\lambda+2\mu+3) + (\mu-\lambda)(\mu+2\lambda+3) \\ &\quad - (\lambda+2\mu+3)(\mu+2\lambda+3)],\end{aligned} \tag{3.127}$$

$$D_3 = -\tfrac{1}{9}(\mu-\lambda)(\lambda+2\mu+3)(\mu+2\lambda+3). \tag{3.128}$$

Hence $\tfrac{1}{3}(\mu-\lambda)$, $-\tfrac{1}{3}(\lambda+2\mu+3)$ and $\tfrac{1}{3}(\mu+2\lambda+3)$ are the roots of the polynomial equation

$$x^3 - \tfrac{1}{2}(D_2+2)\,x - \tfrac{1}{3}D_3 = 0. \tag{3.129}$$

Now using (3.41), (3.42), (3.126) and (3.127) one finds

$$\begin{aligned}-\tfrac{1}{2}(2+D_2) &= -\tfrac{1}{2}(2C_2 + 1 - \tfrac{1}{12}P^2) \\ &= \tfrac{1}{24}(K+L+M)^2 + \tfrac{1}{16}[KL + ML + MK - (K+L+M)^2] \\ &= \tfrac{1}{144}[(2K-L-M)(2L-M-K) + (2K-L-M)(2M-L-K) \\ &\quad + (2L-M-K)(2M-K-L)],\end{aligned} \tag{3.130}$$

$$\begin{aligned}\tfrac{1}{3}D_3 &= \tfrac{1}{64}[(KLM - (KL+LM+KM)(K+L+M)) \\ &\quad + \tfrac{5}{432}(K+L+M)^3 - \tfrac{1}{12}(K+L+M)] \\ &\quad - \tfrac{1}{3}(K+L+M)\{-\tfrac{1}{4} - \tfrac{1}{32}[KL+LM+KM-(K+L+M)^2]\} \\ &= \tfrac{1}{1728}[2(K+L+M)^3 + 27KLM - 9(K+L+M)(KL+LM+KM)] \\ &= \tfrac{1}{1728}(2K-L-M)(2L-K-M)(2M-L-K).\end{aligned} \tag{3.131}$$

Hence $\mu-\lambda$, $-(\lambda+2\mu+3)$ and $\mu+2\lambda+3$ are in some order $\frac{1}{4}(2K-L-M)$, $\frac{1}{4}(2L-K-M)$ and $\frac{1}{4}(2M-L-K)$, which, by using (3.23), (3.24) and (3.25) are in turn, $4d+5+s+2r$, $-2s-r-2d-4$ and $s-r-2d-1$. Since $s \geqslant 0$, $r \geqslant 0$ and $d > -1$, one must have

$$s+2r+5+4d = \mu+2\lambda+3, \tag{3.132}$$

$$s-r-2d-1 = \mu-\lambda, \tag{3.133}$$

$$-2s-r-2d-4 = -\lambda-2\mu-3. \tag{3.134}$$

Hence

$$\mu = s, \tag{3.135}$$

$$\lambda = r+2d+1, \tag{3.136}$$

$$p = s-r+4d+2. \tag{3.137}$$

In particular $2d$ is an integer greater than or equal to minus one.

This is the key extra requirement on the general $\mathscr{R}$-representation needed for it to be possible to build the intertwining SU(3) algebra into unitary SU(3) representations.

With $2d$ integral, the roots of $\phi(x) = 0$ (equation (3.11)) are $\pm j_{\min}$, $\pm(j_{\max}+1)$ and $\pm\alpha$, where all roots are integral or half-integral together and $\alpha \geqslant j_{\max}+1$.

Next note that the entire formalism is symmetric under permutations of K, L, M and P, with a single crucial exception: from (3.20)–(3.25), K, L, M, P and J obey the inequalities

$$\tfrac{1}{4}(K+P) > -1-\tfrac{1}{4}(L+P) \geqslant J \geqslant |\tfrac{1}{4}(M+P)| \tag{3.138}$$

which are reflected in the asymmetric nature of the function $e(K, L, M, P, J)$.

However, under permutations of K, L, M and P it is always possible to adjust the definition of $e(K, L, M, P, J)$ to solve the permuted equation corresponding to (3.82), with one important subtlety: in solving equation (3.82), with P replaced by K or L for an operator r^A which respectively raises K or L by three units, while lowering L, M, P or L, M, K each by one unit, in order for $r^{A'}$ to be the Hermitian conjugate of r^A, one sees from equations (3.82), (3.71), (3.73), (3.74), (3.78) and (3.79) that $e(K, L, M, P, J)$ must change sign if J changes from integral to half-integral. Since no $\mathscr{R}$-representation contains both half-integer and integer spin states, this is no problem, provided the r^A operator is regarded as a mathematical device and not a (complex) observable, since a Hilbert space is still a Hilbert space, if the norm of every non-zero state is negative!

Thus if the half-integer spin states are regarded as separated from the integer spin states, equation (3.82) can be solved for an arbitrary permutation of K, L, M and P.

On the other hand, if the half-integer spin states are placed in the same Hilbert space as the integer spin states, then equation (3.82) can only be solved with M in place of P, but not with K or L in place of P. However, at the same time one sees the remedy: with K or L in place of P one needs a minus sign in front of the $r^A r^{A'}$ term, or equivalently, $r^{A'}$ is to be taken as the negative adjoint of r^A. Then all is well.

Let s^A, t^A and u^A be the solutions of (3.82) with, respectively, K, L or M in place of P, with adjoints, respectively, $-s^{A'}$, $-t^{A'}$ and $u^{A'}$. Then $S^{AA'}$, s^A, $s^{A'}$ and K generate SU(1, 2), as do $S^{AA'}$, t^A, $t^{A'}$ and L, whereas $S^{AA'}$, u^A, $u^{A'}$ and M generate SU(3).

Consider, first, the case where M replaces P. The symmetry carries right through, the intertwining operators have the SU(3) structure and the new Casimir operators, $\mu'-\lambda'$,

$-(\lambda'+2\mu'+3)$ and $\mu'+2\lambda'+3$ are in some order $\frac{1}{4}(2K-L-P)$, $\frac{1}{4}(2L-K-P)$ and $\frac{1}{4}(2P-L-K)$, which are, in turn, $s+2r+2d+4$, $-2s-r-4d-5$ and $s-r+2d+1$. Hence

$$\mu' = s+2d+1, \tag{3.139}$$

$$\lambda' = r, \tag{3.140}$$

$$m = s-r-4d-2. \tag{3.141}$$

Up to now, in either the P formalism or the M formalism, it has only been shown that the intertwining algebra obeys the SU(3) commutation rules. It has not yet been shown that the states do combine together to form complete SU(3) representations. In fact, they do not, yet.

Considering the P formalism, given λ and μ, P must range from $-(2\mu+\lambda)$ to $2\lambda+\mu$ for a full SU(3) representation. If one uses (3.135), (3.136) and (3.137), $2d+1$ would have to range from $-\mu$ to λ, in conflict with the requirement that $2d+1 \geqslant 0$, unless $\mu = 0$. The trouble here is that as one lowers the P value, using the r^A operator, to the point at which $2d+1$ is zero, P and M become indistinguishable, since from (3.25), $2d+1 = \frac{1}{4}(p-m)$. To complete the SU(3) representation, one must switch at this point to the M formalism.

To unify the P and M formalisms explicitly, put $\lambda = \lambda'$ and $\mu = \mu'$, and define b, λ and μ by

$$|b-\tfrac{1}{3}(\mu-\lambda)| = 2d+1, \tag{3.142}$$

$$\lambda+\mu+2 = r+s+2d+3, \tag{3.143}$$

$$b-\tfrac{2}{3}(\lambda-\mu) = s-r. \tag{3.144}$$

Then when $b-\frac{1}{3}(\mu-\lambda)$ is chosen non-negative

$$b = \tfrac{1}{3}(s-r+4d+2) = \tfrac{1}{3}p, \tag{3.145}$$

$$\lambda = r+2d+1, \tag{3.146}$$

$$\mu = s. \tag{3.147}$$

When $b-\frac{1}{3}(\mu-\lambda)$ is chosen non-positive

$$b = \tfrac{1}{3}(s-r-4d-2) = \tfrac{1}{3}m, \tag{3.148}$$

$$\lambda = r, \tag{3.149}$$

$$\mu = s+2d+1, \tag{3.150}$$

Using b, λ and μ to describe the states allows one to treat the M and P formalisms simultaneously. One sees that l^A, m^A, $l^{A'}$ and $m^{A'}$ all leave λ and μ unchanged. So all operators to be considered are (for now) diagonal in λ and μ. Accordingly, states $|b,\lambda,\mu,j,q\rangle$ will be abbreviated by just $|b,j,q\rangle$, with scalar product

$$\langle b,j,q|b,j,q\rangle = e(b,j)\,q!(\bar{\alpha}^D P_{DD'}\alpha^{D'})^{2j}[(2j+1)\,(2j-q)!]^{-1}. \tag{3.151}$$

Then equations (3.31)–(3.34), (3.43), (3.64), (3.65), (3.77), (3.70), (3.71) and (3.72) read

$$l^A|b,j,q\rangle = (\bar{\alpha}^D P_{DD'}\alpha^{D'})^{-1}[\alpha^{B'}P_{B'}{}^A|b-1,j+\tfrac{1}{2},q+1\rangle+(2j+1-q)\,\bar{\alpha}^A|b-1,j+\tfrac{1}{2},q\rangle], \tag{3.152}$$

$$m^A|b,j,q\rangle = P^A{}_{A'}\alpha^{A'}|b-1,j-\tfrac{1}{2},q\rangle - q\bar{\alpha}^A|b-1,j-\tfrac{1}{2},q-1\rangle, \tag{3.153}$$

$$l^{A'}|b,j,q\rangle = \alpha^{A'}|b+1,j-\tfrac{1}{2},q\rangle + qP^{A'}{}_A\bar{\alpha}^A|b+1,j-\tfrac{1}{2},q-1\rangle, \tag{3.154}$$

$$m^{A'}|b,j,q\rangle = (\bar{\alpha}^D P_{DD'}\alpha^{D'})^{-1}[-\alpha^{A'}|b+1,j+\tfrac{1}{2},q+1\rangle+(2j+1-q)\,P^{A'}{}_A\bar{\alpha}^A|b+1,j+\tfrac{1}{2},q\rangle], \tag{3.155}$$

$$2(J+1)\,\delta(B,J+1)\,\gamma(B,J) = [J+1+\tfrac{1}{2}B+\tfrac{1}{3}(2\lambda+\mu+3)]\,[J+1-\tfrac{1}{2}B-\tfrac{1}{3}(2\lambda+\mu+3)] \times [J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\lambda-\mu)]\,[J+1-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda-\mu)] \times [J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\lambda+2\mu+3)]\,[J+1-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda+2\mu+3)], \tag{3.156}$$

$$a(B+1,J+\tfrac{1}{2})\,d(B,J) = \gamma(B,J), \tag{3.157}$$

$$c(B+1,J-\tfrac{1}{2})\,b(B,J) = 2J\delta(B,J), \tag{3.158}$$

$$a(B+1,-1)\,b(B,-\tfrac{1}{2})+c(B+1,0)\,d(B,-\tfrac{1}{2}) = 0, \tag{3.159}$$

$$a(B+1,J-\tfrac{1}{2})\,[e(B+1,J-\tfrac{1}{2})]^{-1} = \bar{b}(B,J)\,[e(B,J)]^{-1}, \tag{3.160}$$

$$c(B+1,J+\tfrac{1}{2})\,[e(B+1,J-\tfrac{1}{2})]^{-1} = \bar{d}(B,J)\,[e(B,J)]^{-1}, \tag{3.161}$$

$$(2J+2)\,\delta(B,J+1)\,[\bar{\gamma}(B,J)]^{-1} = e(B,J+1)\,[e(B,J)]^{-1}. \tag{3.162}$$

When $b-\frac{1}{3}(\mu-\lambda)$ is non-negative, equations (3.73), (3.78) and (3.79) read

$$\begin{aligned}\lambda(J) &= a(B+1,J-\tfrac{1}{2})\,b(B,J)\\ &= -[J-\tfrac{1}{4}(K+P)]\,[J-\tfrac{1}{4}(M+P)]\,[J-\tfrac{1}{4}(L+P)]\\ &= -[J-\tfrac{1}{2}B-\tfrac{1}{3}(2\lambda+\mu+3)]\,[J-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda-\mu)]\,[J-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda+2\mu+3)],\end{aligned} \tag{3.163}$$

$$\begin{aligned}\mu(J) &= [J+1+\tfrac{1}{2}B+\tfrac{1}{3}(2\lambda+\mu+3)]\,[J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\lambda-\mu)]\\ &\quad\times[J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\lambda+2\mu+3)].\end{aligned} \tag{3.164}$$

When $b-\frac{1}{3}(\mu-\lambda)$ is non-positive, equations (3.73), (3.78) and (3.79) read

$$\begin{aligned}\lambda(J) &= a(B+1,J-\tfrac{1}{2})\,b(B,J)\\ &= -[J-\tfrac{1}{4}(K+M)]\,[J-\tfrac{1}{4}(P+M)]\,[J-\tfrac{1}{4}(L+M)]\\ &= -[J-\tfrac{1}{2}B-\tfrac{1}{3}(2\lambda+\mu+3)]\,[J-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda-\mu)]\,[J-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda+2\mu+3)],\end{aligned} \tag{3.165}$$

$$\begin{aligned}\mu(J) &= [J+1+\tfrac{1}{2}B+\tfrac{1}{3}(2\lambda+\mu+3)]\,[J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\lambda-\mu)]\\ &\quad\times[J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\lambda+2\mu+3)],\end{aligned} \tag{3.166}$$

the crucial point being that $\lambda(J)$ and $\mu(J)$ have the same functional form in either the formalism where P is selected out or where M is, allowing this unification of the M and P formalisms.

To solve all equations one may take, from (3.83)–(3.86)

$$a(B,J) = -[J+1-\tfrac{1}{2}B-\tfrac{1}{3}(2\lambda+\mu+3)], \tag{3.167}$$

$$b(B,J) = [J-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda-\mu)]\,[J-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda+2\mu+3)], \tag{3.168}$$

$$c(B,J) = [J+\tfrac{1}{2}B-\tfrac{1}{3}(\lambda-\mu)]\,[J+\tfrac{1}{2}B+\tfrac{1}{3}(2\lambda+\mu+3)], \tag{3.169}$$

$$d(B,J) = -[J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\lambda+2\mu+3)]. \tag{3.170}$$

The inequality (3.138) gives

$$\lambda+\mu-|B-\tfrac{1}{3}(\mu-\lambda)| \geqslant 2J \geqslant |B-\tfrac{2}{3}(\lambda-\mu)|, \tag{3.171}$$

$$-\tfrac{1}{3}(2\mu+\lambda) \leqslant B \leqslant \tfrac{1}{3}(2\lambda+\mu). \tag{3.172}$$

Also $2J$, λ, μ and $3b$ are integers such that $b-\frac{1}{3}(\mu-\lambda)$ is integral, $\mu > 0$, $J-\frac{3}{2}b$ is integral and

$$\begin{aligned}e(B,J) = g(\lambda,\mu)\,[-J-1+\tfrac{1}{2}B+\tfrac{1}{3}(\lambda+2\mu+3)]!\,[J-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda+2\mu+3)]!\\ \times[J-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda-\mu)]!\,[J+\tfrac{1}{2}B+\tfrac{1}{3}(2\lambda+\mu+3)]!\\ \times[-J-1-\tfrac{1}{2}B+\tfrac{1}{3}(\lambda+2\mu+3)]!\,[J+\tfrac{1}{2}B-\tfrac{1}{3}(\lambda-\mu)]!\end{aligned} \tag{3.173}$$

where $g(\lambda,\mu)$ is positive. Note that all the factorials are now of the form $z!$ where $0 \leqslant z$ is integral, so they are individually positive.

The fundamental equation (3.82) now reads

$$R^a = r^A r^{A'} + \tfrac{1}{16}\hbar^2 P^a(3B+2)^2 - \tfrac{1}{4}\hbar S^a(3B+2), \tag{3.174}$$

whereas in (3.66) and (3.58)–(3.61)

$$r^A = \hbar l^A a(B,J)\,(2J+1)^{-1} + \hbar m^A c(B,J)\,(2J+1)^{-1}, \tag{3.175}$$

$$r^{A'} = \hbar l^{A'} b(B,J)\,(2J+1)^{-1} + \hbar m^{A'} d(B,J)\,(2J+1)^{-1}, \tag{3.176}$$

r^A and $r^{A'}$ obey the commutation rules

$$[r^A, r^B] = 0, \quad [r^A, B] = r^A, \quad [r^A, r^{A'}] = \hbar S^a - \hbar^2 \tfrac{3}{2} B P^a, \tag{3.177}$$

and the trace-free, Hermitian matrix operator

$$A_\alpha^{\alpha'} = \begin{pmatrix} B\hbar & r_A \\ r_{A'} & S_{AA'} - \frac{1}{2}B\hbar P_{AA'} \end{pmatrix} \tag{3.178}$$

generates the SU(3) Lie algebra:

$$[A_\alpha^{\alpha'}, A_\beta^{\beta'}] = \hbar(\delta_\beta^{\alpha'} A_\alpha^{\beta'} - \delta_\alpha^{\beta'} A_\beta^{\alpha'}), \tag{3.179}$$

$$A_\alpha^{\alpha'}\,\delta_{\alpha'}^{\alpha} = 0, \quad \bar{A}_\alpha^{\alpha'} = A_\alpha^{\alpha'}, \tag{3.180}$$

$\delta_{\alpha'}^{\alpha}$ and $\delta_\alpha^{\alpha'}$ being given by

$$\delta_{\alpha'}^{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & P^{AA'} \end{pmatrix}, \quad \delta_\alpha^{\alpha'} = \begin{pmatrix} 1 & 0 \\ 0 & P_{AA'} \end{pmatrix} \tag{3.181}$$

(here α and α' are amalgamated indices, which interchange under complex conjugation, $(\cdot, {}^A)$ and $(\cdot, {}^{A'})$ respectively, $\cdot$ being a one-dimensional index).

To end this section consider the possibility of extending the parity operation $\mathscr{p}$ to the r^A operator. This seems to be impossible in the framework established, so far, for reasons which will become clearer, when the description of the r^A operation is discussed in terms of quantum fields. However the operation $\mathscr{cp}$ does extend: with the definitions

$$\mathscr{cp}\, r^A (\mathscr{cp})^{-1} = \alpha t^A{}_{A'} P^{A'}{}_B r^B, \tag{3.182}$$

$$\mathscr{cp}\, r^{A'} (\mathscr{cp})^{-1} = \bar{\alpha} t^{A'}{}_A P^A{}_{B'} r^{B'}, \quad \alpha\bar{\alpha} = 1, \tag{3.183}$$

$$\mathscr{cp}\, B (\mathscr{cp})^{-1} = -B, \tag{3.184}$$

together with, from (1.59), (1.56), (1.51) and (1.52),

$$\mathscr{cp}\, R^a (\mathscr{cp})^{-1} = t^A{}_{B'} R^b t_B{}^{A'}, \tag{3.185}$$

$$\mathscr{cp}\, S^a (\mathscr{cp})^{-1} = -t^A{}_{B'} S^b t_B{}^{A'}, \tag{3.186}$$

$$\mathscr{cp}\, P^a (\mathscr{cp})^{-1} = t^A{}_{B'} P^b t_B{}^{A'}, \tag{3.187}$$

$$\mathscr{cp}\, M^{ab} (\mathscr{cp})^{-1} = t^A{}_{C'} t^{A'}{}_C t^B{}_{D'} t^{B'}{}_D M^{cd}, \tag{3.188}$$

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one obtains an automorphism of all the algebraic structure obtained up to now, especially equations (3.174) and (3.177). The operation $\mathcal{cp}$, of course, reduces just to the ordinary parity operation of the Poincaré sub-algebra.

One notes that the map

$$r^A \rightarrow e^{i\theta B} r^A e^{-i\theta B} = e^{-i\theta} r^A, \tag{3.189}$$

$$r^{A'} \rightarrow e^{i\theta B} r^{A'} e^{-i\theta B} = e^{i\theta} r^{A'}, \tag{3.190}$$

$$P^a, M^{ab}, R^a \rightarrow P^a, M^{ab}, R^a \tag{3.191}$$

also gives an automorphism of the algebra, so by combining $\mathcal{cp}$ with $e^{i\theta B}$ for suitable θ one may choose $\alpha = 1 = \bar{\alpha}$ in (3.182) and (3.183), the general automorphism then being $\mathcal{cp}\, e^{i\phi B}$ for real ϕ.

Note also that the transformation (3.182)–(3.188) may be neatly encapsulated by a transformation on the indices α', α:

$$A^{\alpha'}_{\alpha} \rightarrow T^{\beta}_{\alpha} A^{\beta'}_{\beta} T^{\alpha'}_{\beta'}, \tag{3.192}$$

$$\delta^{\alpha'}_{\alpha} \rightarrow T^{\beta}_{\alpha} \delta^{\beta'}_{\beta} T^{\alpha}_{\beta'}, \quad \delta^{\alpha}_{\alpha'} \rightarrow (T^{-1})^{\alpha}_{\beta} \delta^{\beta}_{\beta'} (T^{-1})^{\beta'}_{\alpha'}, \tag{3.193}$$

where T^{β}_{α} and $T^{\beta'}_{\alpha'}$ are given by

$$T^{\beta}_{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha t_{C'A} P^{C'B} \end{pmatrix}, \tag{3.194}$$

$$T^{\beta'}_{\alpha'} = \begin{pmatrix} 1 & 0 \\ 0 & -\bar{\alpha} P_{CB'} t^{CA'} \end{pmatrix}. \tag{3.195}$$

4. The complete intertwining algebra

In the previous section equation (3.82) was solved and the SU(3) intertwining algebra derived. Now the permuted equations corresponding to the K or L formalisms will be tackled. One requires the solutions to

$$R^a = s^A s^{A'} + \tfrac{1}{16}\hbar^2 P^a (K+2)^2 - \tfrac{1}{4}\hbar S^a (K+2), \tag{4.1}$$

$$R^a = t^A t^{A'} + \tfrac{1}{16}\hbar^2 P^a (L+2)^2 - \tfrac{1}{4}\hbar S^a (L+2), \tag{4.2}$$

where $\bar{s}^{A'} = -s^{A'}$, $t^{A'} = -\bar{t}^{A'}$, in accordance with the discussion in § 3. The work will be done in the P formalism, by using the scalar product and the values of $\gamma(K, L, M, P, J)$ and $\delta(K, L, M, P\, J)$ given in equations (3.89), (3.87) and (3.88). The transformation s^A will take (K, L, M, P) to $(K-3, L+1, M+1, P+1)$, t^A will take (K, L, M, P) to $(K+1,\ L-3,\ M+1,\ P+1)$.

First consider s^A. Let l^A_s, m^A_s, $l^{A'}_s$, $m^{A'}_s$ be the appropriately permuted operators corresponding to l^A, m^A, $l^{A'}$ and $m^{A'}$ given in equations (3.31)–(3.34). Then

$$s^A = \hbar l^A_s f(K, L, M, P, J)\,(2J+1)^{-1} + \hbar m^A_s h(K, L, M, P, J)\,(2J+1)^{-1}, \tag{4.3}$$

$$s^{A'} = \hbar l^{A'}_s g(K, L, M, P, J)\,(2J+1)^{-1} + \hbar m^{A'}_s k(K, L, M, P, J)\,(2J+1)^{-1}, \tag{4.4}$$

where $f(K, L, M, P, J)$, $h(K, L, M, P, J)$, $g(K, L, M, P, J)$ and $k(K, L, M, P, J)$ are to be found.

Then the analogues of (3.70) and (3.71) are

$$\begin{aligned} f(K+3, L-1, M-1, P-1, J-\tfrac{1}{2})\,[\bar{g}(K, L, M, P, J)]^{-1} \\ = -e(K+3, L-1, M-1, P-1, J-\tfrac{1}{2})\,[e(K, L, M, P, J)]^{-1} \\ = -[J-\tfrac{1}{4}(K+M)]\,[J-\tfrac{1}{4}(K+P)]\,[J-\tfrac{1}{4}(K+L)]^{-1}, \end{aligned} \tag{4.5}$$

$$h(K+3,L-1,M-1,P-1,J+\tfrac{1}{2})\,[\bar{k}(K,L,M,P,J)]^{-1}$$
$$= -e(K+3,L-1,M-1,P-1,J+\tfrac{1}{2})\,[e(K,L,M,P,J)]^{-1}$$
$$= [J+1+\tfrac{1}{4}(K+L)]\,[J+1+\tfrac{1}{4}(K+M)]\,[J+1+\tfrac{1}{4}(K+P)]. \tag{4.6}$$

The analogues of equations (3.64) and (3.65) are

$$f(K+3,L-1,M-1,P-1,J+\tfrac{1}{2})\,k(K,L,M,P,J) = \gamma(K,L,M,P,J)$$
$$= [J+1-\tfrac{1}{4}(K+M)]\,[J+1-\tfrac{1}{4}(K+P)], \tag{4.7}$$

$$h(K+3,L-1,M-1,P-1,J-\tfrac{1}{2})\,g(K,L,M,P,J) = 2J\delta(K,L,M,P,J)$$
$$= [J-\tfrac{1}{4}(K+L)]\,[J+\tfrac{1}{4}(K+L)]\,[J+\tfrac{1}{4}(K+M)]\,[J+\tfrac{1}{4}(K+P)]. \tag{4.8}$$

Here (3.87), (3.88) and (3.89) have been used, with $g(K-L, L-M)$ taken to be just $[(K-L)!]^2$ in (3.89).

Equations (4.5)–(4.8) are solved by

$$f(K,L,M,P,J) = -[J+1-\tfrac{1}{4}(K+M)]\,[J+1-\tfrac{1}{4}(K+P)], \tag{4.9}$$

$$g(K,L,M,P,J) = J-\tfrac{1}{4}(K+L), \tag{4.10}$$

$$h(K,L,M,P,J) = [J+\tfrac{1}{4}(K+L)]\,[J+\tfrac{1}{4}(K+M)]\,[J+\tfrac{1}{4}(K+P)], \tag{4.11}$$

$$k(K,L,M,P,J) = 1. \tag{4.12}$$

Then instead of equations (3.73) (3.74), (3.78) and (3.79), one has

$$\lambda(J) = f(K+3,L-1,M-1,P-1,J-\tfrac{1}{2})\,g(K,L,M,P,J)$$
$$= -[J-\tfrac{1}{4}(K+L)]\,[J-\tfrac{1}{4}(K+M)]\,[J-\tfrac{1}{4}(K+L)], \tag{4.13}$$

$$\mu(J) = h(K+3,L-1,M-1,P-1,J+\tfrac{1}{2})\,k(K,L,M,P,J)$$
$$= -[J+1+\tfrac{1}{4}(K+L)]\,[J+1+\tfrac{1}{4}(K+M)]\,[J+1+\tfrac{1}{4}(K+P)]. \tag{4.14}$$

But (4.13) and (4.14) exactly coincide with the permuted versions of (3.78) and (3.79). Hence equation (4.1) follows.

Now consider t^A. Let l_t^A, m_t^A, $l_t^{A'}$, $m_t^{A'}$ be the permuted operators corresponding to l^A, m^A, $l^{A'}$ and $m^{A'}$ given in equations (3.31)–(3.34). Then

$$t^A = \hbar l_t^A\,u(K,L,M,P,J)\,(2J+1)^{-1} + \hbar m_t^A\,w(K,L,M,P,J)\,(2J+1)^{-1}, \tag{4.15}$$

$$t^{A'} = \hbar l_t^{A'}\,v(K,L,M,P,J)\,(2J+1)^{-1} + \hbar m_t^{A'}\,x(K,L,M,P,J)\,(2J+1)^{-1}. \tag{4.16}$$

The analogues of equations (4.5) to (4.8) read

$$u(K-1,L+3,M-1,P-1,J-\tfrac{1}{2})\,[\bar{v}(K,L,M,P,J)]^{-1}$$
$$= -e(K-1,L+3,M-1,P-1,J-\tfrac{1}{2})\,[e(K,L,M,P,J)]^{-1}$$
$$= -\{[J-\tfrac{1}{4}(L+P)]\,[J-\tfrac{1}{4}(L+K)]\,[J-\tfrac{1}{4}(L+M)]\}^{-1}, \tag{4.17}$$

$$w(K-1,L+3,M-1,P-1,J+\tfrac{1}{2})\,[\bar{x}(K,L,M,P,J)]^{-1}$$
$$= -e(K-1,L+3,M-1,P-1,J+\tfrac{1}{2})\,[e(K,L,M,P,J)]^{-1}$$
$$= -[J+1+\tfrac{1}{4}(L+K)]\,\{[J+1+\tfrac{1}{4}(L+M)]\,[J+1+\tfrac{1}{4}(L+P)]\}^{-1}, \tag{4.18}$$

$$u(K-1,L+3,M-1,P-1,J+\tfrac{1}{2})\,x(K,L,M,P,J) = \gamma(K,L,M,P,J)$$
$$= [J+1+\tfrac{1}{4}(L+M)]\,[J+1+\tfrac{1}{4}(L+P)], \tag{4.19}$$

$$w(K-1,L+3,M-1,P-1,J-\tfrac{1}{2})\,v(K,L,M,P,J) = 2J\delta(K,L,M,P,J)$$
$$= [J-\tfrac{1}{4}(L+K)]\,[J-\tfrac{1}{4}(L+M)]\,[J+\tfrac{1}{4}(L+K)]\,[J-\tfrac{1}{4}(L+P)]. \qquad (4.20)$$

Equations (4.17)–(4.20) are solved by

$$u(K,L,M,P,J) = -1, \qquad (4.21)$$

$$v(K,L,M,P,J) = [J-\tfrac{1}{4}(L+P)]\,[J-\tfrac{1}{4}(L+K)]\,[J-\tfrac{1}{4}(L+M)], \qquad (4.22)$$

$$w(K,L,M,P,J) = J+\tfrac{1}{4}(L+M), \qquad (4.23)$$

$$x(K,L,M,P,J) = -[J+1+\tfrac{1}{4}(L+M)]\,[J+1+\tfrac{1}{4}(L+P)]. \qquad (4.24)$$

Then instead of (4.13) and (4.14) one has

$$\lambda(J) = u(K-1,L+3,M-1,P-1,J-\tfrac{1}{2})\,v(K,L,M,P,J)$$
$$= -[J-\tfrac{1}{4}(L+P)]\,[J-\tfrac{1}{4}(L+K)]\,[J-\tfrac{1}{4}(L+M)], \qquad (4.25)$$

$$\mu(J) = -[J+1+\tfrac{1}{4}(L+K)]\,[J+1+\tfrac{1}{4}(L+M)]\,[J+\tfrac{1}{4}(L+P)] \qquad (4.26)$$

exactly coinciding with the permuted versions of equations (3.78), (3.79), (4.13) and (4.14). Hence equation (4.2) follows.

At this point a summary is in order. One has found r^A, s^A and t^A, with their adjoints, $r^{A'}$, $-s^{A'}$ and $-t^{A'}$. Define in the P formalism ρ, $\tilde{\rho}$, σ, $\tilde{\sigma}$ by

$$\rho|k, l,m,p,j,q\rangle \equiv |k-4,l,m,p+4,j,q\rangle, \qquad (4.27)$$

$$\tilde{\rho}|k,l,m,p,j,q\rangle \equiv |k+4,l,m,p-4,j,q\rangle, \qquad (4.28)$$

$$\sigma|k,l,m,p,j,q\rangle \equiv |k,l-4,m,p+4,j,q\rangle, \qquad (4.29)$$

$$\tilde{\sigma}|k,l,m,p,j,q\rangle \equiv |k,l+4,m,p-4,j,q\rangle. \qquad (4.30)$$

In the (b, λ, μ) formalism, these definitions read

$$\rho|b,\lambda,\mu,j,q\rangle \equiv |b+\tfrac{4}{3},\lambda-1,\mu,j,q\rangle, \qquad (4.31)$$

$$\tilde{\rho}|b,\lambda,\mu,j,q\rangle \equiv |b-\tfrac{4}{3},\lambda+1,\mu,j,q\rangle, \qquad (4.32)$$

$$\sigma|b,\lambda,\mu,j,q\rangle \equiv |b+\tfrac{4}{3},\lambda,\mu+1,j,q\rangle, \qquad (4.33)$$

$$\tilde{\sigma}|b,\lambda,\mu,j,q\rangle \equiv |b-\tfrac{4}{3},\lambda,\mu-1,j,q\rangle. \qquad (4.34)$$

Then in the P formalism, r^A, s^A, t^A, $r^{A'}$, $s^{A'}$ and $t^{A'}$ are given by

$$r^A = \hbar l^A a(K,L,M,P,J)\,(2J+1)^{-1} + \hbar m^A c(K,L,M,P,J)\,(2J+1)^{-1}, \qquad (4.35)$$

$$s^A = \hbar\rho l^A f(K,L,M,P,J)\,(2J+1)^{-1} + \hbar\rho m^A h(K,L,M,P,J)\,(2J+1)^{-1}, \qquad (4.36)$$

$$t^A = \hbar\sigma l^A u(K,L,M,P,J)\,(2J+1)^{-1} + \hbar\sigma m^A w(K,L,M,P,J)\,(2J+1)^{-1}, \qquad (4.37)$$

$$r^{A'} = \hbar l^{A'} b(K,L,M,P,J)\,(2J+1)^{-1} + \hbar m^{A'} d(K,L,M,P,J)\,(2J+1)^{-1}, \qquad (4.38)$$

$$s^{A'} = \hbar\tilde{\rho} l^{A'} g(K,L,M,P,J)\,(2J+1)^{-1} + \hbar\tilde{\rho} m^{A'} k(K,L,M,P,J)\,(2J+1)^{-1}, \qquad (4.39)$$

$$t^{A'} = \hbar\tilde{\sigma} l^{A'} v(K,L,M,P,J)\,(2J+1)^{-1} + \hbar\tilde{\sigma} m^{A'} x(K,L,M,P,J)\,(2J+1)^{-1}, \qquad (4.40)$$

where a, b, c, d, f, g, h, k, u, v, w and x are given in equations (3.83)–(3.86), (4.9)–(4.12) and (4.21)–(4.24). In the (b, λ, μ) formalism, they are given by

$$a = -[J+1-\tfrac{1}{2}B-\tfrac{1}{3}(2\Lambda+M+3)], \tag{4.41}$$

$$b = [J-\tfrac{1}{2}B+\tfrac{1}{3}(\Lambda-M)]\,[J-\tfrac{1}{2}B+\tfrac{1}{3}(\Lambda+2M+3)], \tag{4.42}$$

$$c = [J+\tfrac{1}{2}B-\tfrac{1}{3}(\Lambda-M)]\,[J+\tfrac{1}{2}B+\tfrac{1}{3}(\Lambda+2M+3)], \tag{4.43}$$

$$d = -[J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\Lambda+2M+3)], \tag{4.44}$$

$$f = -[J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\Lambda+2M+3)]\,[J+1-\tfrac{1}{2}B-\tfrac{1}{3}(2\Lambda+M+3)], \tag{4.45}$$

$$g = J+\tfrac{1}{2}B-\tfrac{1}{3}(\Lambda-M), \tag{4.46}$$

$$h = [J-\tfrac{1}{2}B+\tfrac{1}{3}(\Lambda-M)]\,[J-\tfrac{1}{2}B+\tfrac{1}{3}(\Lambda+2M+3)]\,[J+\tfrac{1}{2}B+\tfrac{1}{3}(2\Lambda+M+3)], \tag{4.47}$$

$$k = -1, \tag{4.48}$$

$$u = -1, \tag{4.49}$$

$$v = [J-\tfrac{1}{2}B+\tfrac{1}{3}(\Lambda+2M+3)]\,[J+\tfrac{1}{2}B-\tfrac{1}{3}(\Lambda-M)]\,[J+\tfrac{1}{2}B+\tfrac{1}{3}(2\Lambda+M+3)], \tag{4.50}$$

$$w = [J-\tfrac{1}{2}B+\tfrac{1}{3}(\Lambda-M)], \tag{4.51}$$

$$x = -[J+1-\tfrac{1}{2}B-\tfrac{1}{3}(2\Lambda+M+3)]\,[J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\Lambda+2M+3)]. \tag{4.52}$$

The operators l^A, m^A, $l^{A'}$ and $m^{A'}$ are given in the P-formalism by equations (3.31)–(3.34) and in the (b, λ, μ) formalism by equations (3.152)–(3.155). The scalar product in the P-formalism is given by equation (3.89) with $g(K-L, L-M) = [(K-L)!]^2$ in the (b, λ, μ) formalism and by equation (3.173) with $g(\lambda, \mu) = 1$.

Using the properties of l^A, m^A, $l^{A'}$ and $m^{A'}$ given by equations (3.46)–(3.53) and the relations (4.27)–(4.40), (3.83)–(3.86), (4.9)–(4.12) and (4.21)–(4.24), one finds in the P formalism, the following commutators:

$$[r^A, s^{A'}] = [r^A, t^{A'}] = [s^A, t^{A'}] = 0. \tag{4.53}$$

Considering next, for instance, the commutator of s^A and t^B:

$$\begin{aligned}\hbar^{-2}[s^A, t^B] = {} & -\rho\sigma l^A m^B[J-\tfrac{1}{4}(K+M)]\,[J-\tfrac{1}{4}(K+P)]\,[J+\tfrac{1}{4}(L+K)]\,[2J(2J+1)]^{-1} \\ & +\rho\sigma l^B m^A[J+\tfrac{1}{4}(K+L)]\,[J+\tfrac{1}{4}(K+M)]\,[J+\tfrac{1}{4}(K+P)]\,[2J(2J+1)]^{-1} \\ & -\rho\sigma m^A l^B[J+\tfrac{1}{4}(K+L)]\,[J+1+\tfrac{1}{4}(K+M)]\,[J+\tfrac{1}{4}(K+P)]\,[2J(2J+1)]^{-1} \\ & +\rho\sigma m^B l^A[J+\tfrac{1}{4}(L+K)]\,[J+1-\tfrac{1}{4}(K+M)]\,[J+1-\tfrac{1}{4}(K+P)]\,[(2J+2)\,(2J+1)]^{-1}.\end{aligned} \tag{4.54}$$

Using (3.50) and (3.51) one sees immediately that symmetric parts cancel. Collecting together the skew parts, one obtains just

$$\hbar^{-2}[s^A, t^B]\,|k, l, m, p, j, q\rangle = -\epsilon^{AB}[j+\tfrac{1}{4}(l+k)]\,|k-2, l-2, m+2, p+2, j, q\rangle. \tag{4.55}$$

Similarly one obtains

$$\begin{aligned}&\hbar^{-2}[r^A, s^B]\,|k, l, m, p, j, q\rangle \\ &\qquad = -\epsilon^{AB}[j+1-\tfrac{1}{4}(k+p)]\,[j+\tfrac{1}{4}(k+p)]\,|k-2, l+2, m+2, p-2, j, q\rangle,\end{aligned} \tag{4.56}$$

$$\begin{aligned}&\hbar^{-2}[t^A, r^B]\,|k, l, m, p, j, q\rangle \\ &\qquad = -\epsilon^{AB}|k+2, l-2, m+2, p-2, j, q\rangle.\end{aligned} \tag{4.57}$$

Accordingly, define X, Y and Z in the P-formalism and (b, λ, μ) formalism:

$$X|k,l,m,p,j,q\rangle \equiv [j+\tfrac{1}{4}(k+l)]\,|k-2,l-2,\ m+2,\ p+2,j,q\rangle, \tag{4.58}$$

$$Y|k,l,m,p,j,q\rangle \equiv |k+2,l-2,\ m+2,\ p-2,\ j,q\rangle, \tag{4.59}$$

$$Z|k,l,m,p,j,q\rangle \equiv [j+1-\tfrac{1}{4}(k+p)]\,[j+\tfrac{1}{4}(k+p)]\,|k-2,l+2,\ m+2,\ p-2,j,q\rangle, \tag{4.60}$$

$$X|b,\lambda,\mu,j,q\rangle \equiv [j-\tfrac{1}{2}b+\tfrac{1}{3}(\lambda-\mu)]\,|b+\tfrac{2}{3},\lambda-1,\mu+1,j,q\rangle, \tag{4.61}$$

$$Y|b,\lambda,\mu,j,q\rangle \equiv |b-\tfrac{2}{3},\lambda,\mu+1,j,q\rangle, \tag{4.62}$$

$$Z|b,\lambda,\mu,j,q\rangle \equiv [j+1-\tfrac{1}{2}b-\tfrac{1}{3}(2\lambda+\mu+3)]\,[j+\tfrac{1}{2}b+\tfrac{1}{3}(2\lambda+\mu+3)]\,|b-\tfrac{2}{3},\lambda-1,\mu,j,q\rangle. \tag{4.63}$$

Then X, Y and Z have adjoints $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ given by

$$\bar{X}|k,l,m,p,j,q\rangle = [j-\tfrac{1}{4}(k+l)]\,|k+2,l+2,m-2,p-2,j,q\rangle, \tag{4.64}$$

$$\bar{Y}|k,l,m,p,j,q\rangle = -[j-\tfrac{1}{4}(l+p)]\,[j+1+\tfrac{1}{4}(l+p)]\,|k-2,l+2,m-2,p+2,j,q\rangle, \tag{4.65}$$

$$\bar{Z}|k,l,m,p,j,q\rangle = -|k+2,l-2,m-2,p+2,j,q\rangle, \tag{4.66}$$

$$\bar{X}|b,\lambda,\mu,j,q\rangle = [j+\tfrac{1}{2}b-\tfrac{1}{3}(\lambda-\mu)]\,|b-\tfrac{2}{3},\lambda+1,\mu-1,j,q\rangle, \tag{4.67}$$

$$\bar{Y}|b,\lambda,\mu,j,q\rangle = -[j-\tfrac{1}{2}b+\tfrac{1}{3}(\lambda+2\mu+3)]\,[j+1+\tfrac{1}{2}b-\tfrac{1}{3}(\lambda+2\mu+3)]\,|b+\tfrac{2}{3},\lambda,\mu-1,j,q\rangle, \tag{4.68}$$

$$\bar{Z}|b,\lambda,\mu,j,q\rangle = -|b+\tfrac{2}{3},\lambda+1,\mu,j,q\rangle. \tag{4.69}$$

Then from (4.55)–(4.57) one has

$$[r^A, s^B] = -\hbar^2\epsilon^{AB}Z, \quad [t^A, r^B] = -\hbar^2\epsilon^{AB}Y, \quad [s^A, t^B] = -\hbar^2\epsilon^{AB}X. \tag{4.70}$$

Next, by applying the Jacobi identity and (4.70)

$$\begin{aligned} 0 &= [r^A,[t^B,r^C]]+[r^C,[r^A,t^B]]+[t^B,[r^B,r^A]] \\ &= [r^A, Y]\,(-\hbar^2)\,\epsilon^{BC}+[r^C, Y]\,(-\hbar^2)\,\epsilon^{AB} \\ &= \hbar^2\epsilon^{AC}[r^B, Y]. \end{aligned} \tag{4.71}$$

Hence, and similarly,

$$0 = [r^B, Y] = [r^B, Z], \tag{4.72}$$

$$0 = [s^B, Z] = [s^B, X], \tag{4.73}$$

$$0 = [t^B, X] = [t^B, Y]. \tag{4.74}$$

Again, from the Jacobi identity, if λ^A, μ^A and ν^A are defined by

$$[r^A, X] \equiv \lambda^A, \quad [s^A, Y] \equiv \mu^A, \quad [t^A, Z] \equiv \nu^A \tag{4.75}$$

then

$$\begin{aligned} 0 &= [s^A,[t^B,r^C]]+[r^B,[t^C,s^A]]+[t^C,[s^A,r^B]] \\ &= -\hbar^2(\mu^A\epsilon^{BC}+\lambda^B\epsilon^{CA}+\nu^C\epsilon^{AB}). \end{aligned} \tag{4.76}$$

Hence

$$\lambda^A = \mu^A = \nu^A. \tag{4.77}$$

Define τ and $\tilde{\tau}$ in the P-formalism by

$$\tau|k,l,m,p,j,q\rangle \equiv |k-2,l-2,m+2,p+2,j,q\rangle, \tag{4.78}$$

$$\tilde{\tau}|k,l,m,p,j,q\rangle \equiv |k+2,l+2,m-2,p-2,j,q\rangle, \tag{4.79}$$

and, in the (b, λ, μ)-formalism, by

$$\tau|b, \lambda, \mu, j, q\rangle \equiv |b+\tfrac{2}{3}, \lambda-1, \mu+1, j, q\rangle, \tag{4.80}$$

$$\tilde{\tau}|b, \lambda, \mu, j, q\rangle \equiv |b-\tfrac{2}{3}, \lambda+1, \mu-1, j, q\rangle. \tag{4.81}$$

Then computing $[r^A, X] = \lambda^A$, one finds

$$\lambda^A = \hbar\tau l^A l(K, L, M, P, J)\,(2J+1)^{-1} + \hbar\tau m^A n(K, L, M, P, J)\,(2J+1)^{-1}, \tag{4.82}$$

$$\lambda^{A'} = \hbar\tilde{\tau} l^{A'} m(K, L, M, P, J)\,(2J+1)^{-1} + \hbar\tilde{\tau} m^{A'} p(K, L, M, P, J)\,(2J+1)^{-1}, \tag{4.83}$$

where $\lambda^{A'} = \bar{\lambda}^{A'}$ and, in the P formalism,

$$l(K, L, M, P, J) = J+1-\tfrac{1}{4}(K+P), \tag{4.84}$$

$$m(K, L, M, P, J) = -[J-\tfrac{1}{4}(K+L)]\,[J-\tfrac{1}{4}(L+P)], \tag{4.85}$$

$$n(K, L, M, P, J) = [J+\tfrac{1}{4}(K+L)]\,[J+\tfrac{1}{4}(K+P)], \tag{4.86}$$

$$p(K, L, M, P, J) = -[J+1+\tfrac{1}{4}(L+P)], \tag{4.87}$$

and, in the (b, λ, μ) formalism,

$$l = J+1-\tfrac{1}{2}B-\tfrac{1}{3}(\Lambda+2M+3), \tag{4.88}$$

$$m = -[J+\tfrac{1}{2}B-\tfrac{1}{3}(\Lambda-M)]\,[J-\tfrac{1}{2}B+\tfrac{1}{3}(\Lambda+2M+3)], \tag{4.89}$$

$$n = [J-\tfrac{1}{2}B+\tfrac{1}{3}(\Lambda-M)]\,[J+\tfrac{1}{2}B+\tfrac{1}{3}(2\Lambda+M+3)], \tag{4.90}$$

$$p = -[J+1+\tfrac{1}{2}B-\tfrac{1}{3}(\Lambda+2M+3)]. \tag{4.91}$$

Using (4.53), (4.70) and the Jacobi identity, one obtains quickly

$$0 = [r^{A'}, X] = [s^{A'}, Y] = [t^{A'}, Z]. \tag{4.92}$$

Using (3.91), (4.53), (4.70), the relations

$$[t^B, P] = -t^B, \quad [t^B, S^{AA'}] = -\hbar t^A P^{BA'} + \tfrac{1}{2}\hbar t^B P^{AA'} \tag{4.93}$$

(analogous to (3.93) and (3.96)), and the Jacobi identity, one has again

$$\begin{aligned} -\hbar^2[r^{A'}, Y]\,\epsilon^{BC} &= [r^{A'}, [t^B, r^C]] \\ &= [t^B, [r^{A'}, r^C]] = [t^B, -\hbar S^{CA'} + \tfrac{1}{2}\hbar^2 P P^{CA'}] \\ &= \hbar^2 t^C P^{BA'} - \hbar^2 t^B P^{CA'} = \hbar^2 \epsilon^{BC} t^D P_D{}^{A'}. \end{aligned} \tag{4.94}$$

Hence, and similarly,

$$[r^{A'}, Y] = -t^D P_D{}^{A'}, \tag{4.95}$$

$$[r^{A'}, Z] = -s^D P_D{}^{A'}, \tag{4.96}$$

$$[s^{A'}, Z] = -r^D P_D{}^{A'}, \tag{4.97}$$

$$[s^{A'}, X] = -t^D P_D{}^{A'}, \tag{4.98}$$

$$[t^{A'}, X] = -s^D P_D{}^{A'}, \tag{4.99}$$

$$[t^{A'}, Y] = -r^D P_D{}^{A'}. \tag{4.100}$$

By using (4.75), (4.77), (4.70) and the Jacobi identity,

$$\begin{aligned} [\lambda^A, t^B] &= [[s^A, Y], t^B] = [[s^A, t^B], Y] = -\hbar^2[X, Y]\,\epsilon^{AB} \\ &= [X, [t^A, r^B]] = [t^A, [X, r^B]] = [\lambda^B, t^A] = 0. \end{aligned} \tag{4.101}$$

Hence, and similarly,
$$[X, Y] = [Y, Z] = [Z, X] = 0, \tag{4.102}$$
$$[\lambda^A, t^B] = [\lambda^A, s^B] = [\lambda^A, r^B] = 0. \tag{4.103}$$

By using (4.75), (4.53), (4.98) and (4.70),
$$[\lambda^A, s^{B'}] = [[r^A, X], s^{B'}] = [[s^{B'}, X], r^A]$$
$$= -[t^D P_D{}^{B'}, r^A] = -\hbar^2 Y P^{AB'}, \tag{4.104}$$
$$[\lambda^A, t^{B'}] = -\hbar^2 Z P^{AB'}, \tag{4.105}$$
$$[\lambda^A, r^{B'}] = -\hbar^2 X P^{AB'}. \tag{4.106}$$

By using (4.70), (4.92), (4.99), (4.53),
$$-\hbar^2 [X, \bar{Y}]\, \epsilon^{A'B'} = -[X, [r^{B'}, t^{A'}]] = -[r^{B'}, [X, t^{A'}]] = 0. \tag{4.107}$$

So, and similarly,
$$[X, \bar{Y}] = [X, \bar{Z}] = [Y, \bar{Z}] = 0. \tag{4.108}$$

By using (4.70), (4.98), (4.99) and the analogues of (3.91) for s^A and t^A,
$$-\hbar^2 \epsilon^{A'B'} [X, \bar{X}] = [X, [t^{B'}, s^{A'}]]$$
$$= [t^{B'}, [X, s^{A'}]] + [s^{A'}, [t^{B'}, X]]$$
$$= [t^{B'}, t^D P_D{}^{A'}] - [s^{A'}, s^D P_D{}^{A'}]$$
$$= \tfrac{1}{2}\hbar^2 \epsilon^{A'B'} (L + K). \tag{4.109}$$

So, and similarly,
$$[X, \bar{X}] = -\tfrac{1}{2}(L + K), \tag{4.110}$$
$$[\bar{Y}, \bar{Y}] = \tfrac{1}{2}(L + P), \tag{4.111}$$
$$[Z, \bar{Z}] = \tfrac{1}{2}(K + P). \tag{4.112}$$

Using (4.75), (4.108), (4.95) and (4.99), one has
$$[\lambda^{A'}, Y] = -[[r^{A'}, \bar{X}], Y] = -[[r^{A'}, Y], \bar{X}]$$
$$= [t^D P_D{}^{A'}, \bar{X}] = -\overline{[X, t^{D'} P_{D'}{}^{A}]}$$
$$= -\overline{s^B P_B{}^{D'} P_{D'}{}^{A}}$$
$$= -s^{A'}, \tag{4.113}$$
$$[\lambda^{A'}, Z] = -t^{A'}, \tag{4.114}$$
$$[\lambda^{A'}, X] = -r^{A'}. \tag{4.115}$$

By using (4.75), (4.73) and (4.102)
$$[\lambda^A, X] = [[s^A, Y], X] = [[X, Y], s^A] = 0, \tag{4.116}$$
$$[\lambda^A, Y] = 0, \tag{4.117}$$
$$[\lambda^A, Z] = 0, \tag{4.118}$$

By using (4.75), (4.117) and (4.103)
$$[\lambda^A, \lambda^B] = [\lambda^A, [s^B, Y]] = 0. \tag{4.119}$$

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Finally, using (4.75), (4.113), (4.104), (4.111) and the analogue of (3.91) for s^A,

$$\begin{aligned}[\lambda^{A'}, \lambda^A] &= [\lambda^{A'}, [s^A, Y]] \\ &= [s^A, [\lambda^{A'}, Y]] + [Y, [s^A, \lambda^{A'}]] \\ &= -[s^A, s^{A'}] + [Y, \overline{[s^{A'}, \lambda^A]}] \\ &= -[s^A, s^{A'}] + \hbar^2 [Y, \overline{Y}]\, P^{AA'} \\ &= -\hbar(S^a - \tfrac{1}{2}\hbar^2 K P^a) + \tfrac{1}{2}\hbar^2 P^a (L+P) \\ &= -\hbar(S^a + \tfrac{1}{2}\hbar^2 M P^a). \end{aligned} \tag{4.120}$$

Equations (4.53)–(4.120) establish that S^a, K, L, M, P, r^A, s^A, t^A, λ^A, X, Y and Z form a real Lie algebra of twenty-eight dimensions, three for S^a, three for K, L, M and P, since $K+L+M+P = 0$, four for each of r^A, s^A, t^A and λ^A and two for each of X, Y and Z, giving twenty-eight in all. The full result will now be summarized in the (b, λ, μ) formalism:

THEOREM B. *The operators r^A, s^A, and t^A with their adjoints $r^{A'}$, $-s^{A'}$, $-t^{A'}$ generate a twenty-eight dimensional real Lie algebra $\mathfrak{Q}$ under commutation, spanned by S^a, B, $(A+1)$, $(M+1)$, r^A, s^A, t^A, $r^{A'}$, $s^{A'}$, $t^{A'}$, X, Y, Z, $\overline{X}$, $\overline{Y}$, and $\overline{Z}$, obeying the following commutation rules*:

$$[S^a, S^b] = i\hbar\epsilon^{abcd} P_c S_d, \tag{4.121}$$

$$[S^b, r^A] = \hbar r^B P^{AB'} - \tfrac{1}{2}\hbar r^A P^{BB'}, \tag{4.122}$$

$$[S^b, s^A] = \hbar s^B P^{AB'} - \tfrac{1}{2}\hbar s^A P^{BB'}, \tag{4.123}$$

$$[S^b, t^A] = \hbar t^B P^{AB'} - \tfrac{1}{2}\hbar t^A P^{BB'}, \tag{4.124}$$

$$[S^b, \lambda^A] = \hbar\lambda^B P^{AB'} - \tfrac{1}{2}\hbar\lambda^A P^{BB'}, \tag{4.125}$$

$$[r^A, s^B] = -\hbar^2\epsilon^{AB} Z, \tag{4.126}$$

$$[s^A, t^B] = -\hbar^2\epsilon^{AB} X, \tag{4.127}$$

$$[t^A, r^B] = -\hbar^2\epsilon^{AB} Y, \tag{4.128}$$

$$[\lambda^A, r^B] = 0, \tag{4.129}$$

$$[\lambda^A, s^B] = 0, \tag{4.130}$$

$$[\lambda^A, t^B] = 0, \tag{4.131}$$

$$[X, r^A] = -\lambda^A, \tag{4.132}$$

$$[X, s^A] = 0, \tag{4.133}$$

$$[X, t^A] = 0, \tag{4.134}$$

$$[X, \lambda^A] = 0, \tag{4.135}$$

$$[Y, r^A] = 0, \tag{4.136}$$

$$[Y, s^A] = -\lambda^A, \tag{4.137}$$

$$[Y, t^A] = 0, \tag{4.138}$$

$$[Y, \lambda^A] = 0, \tag{4.139}$$

$$[Z, r^A] = 0, \tag{4.140}$$

$$[Z, s^A] = 0, \tag{4.141}$$

$$[Z, t^A] = -\lambda^A, \tag{4.142}$$

$$[Z, \lambda^A] = 0, \tag{4.143}$$

$$[X, S^a] = 0, \tag{4.144}$$

$$[Y, S^a] = 0, \tag{4.145}$$

$$[Z, S^a] = 0, \tag{4.146}$$

$$[r^A, r^B] = 0, \tag{4.147}$$

$$[s^A, s^B] = 0, \tag{4.148}$$

$$[t^A, t^B] = 0, \tag{4.149}$$

$$[\lambda^A, \lambda^B] = 0, \tag{4.150}$$

$$[r^A, r^{A'}] = \hbar S^a - \tfrac{1}{2}\hbar^2 P^a(3B), \tag{4.151}$$

$$[r^A, s^{A'}] = 0, \tag{4.152}$$

$$[r^A, t^{A'}] = 0, \tag{4.153}$$

$$[r^A, \lambda^{A'}] = -\hbar^2 \bar{X} P^a, \tag{4.154}$$

$$[s^A, s^{A'}] = \hbar S^a - \tfrac{1}{2}\hbar^2 P^a[-3B + \tfrac{4}{3}(2\Lambda + M + 3)], \tag{4.155}$$

$$[s^A, t^{A'}] = 0, \tag{4.156}$$

$$[s^A, \lambda^{A'}] = \hbar^2 \bar{Y} P^a, \tag{4.157}$$

$$[t^A, t^{A'}] = \hbar S^a - \tfrac{1}{2}\hbar^2 P^a[-B - \tfrac{4}{3}(\Lambda + 2M + 3)], \tag{4.158}$$

$$[t^A, \lambda^{A'}] = \hbar^2 \bar{Z} P^a, \tag{4.159}$$

$$[\lambda^A, \lambda^{A'}] = \hbar S^a + \tfrac{1}{2}\hbar^2 P^a[\tfrac{4}{3}(M - \Lambda) - B], \tag{4.160}$$

$$[r^A, \bar{X}] = 0, \tag{4.161}$$

$$[r^A, \bar{Y}] = -t^{D'} P_{D'}{}^A, \tag{4.162}$$

$$[r^A, \bar{Z}] = -s^{D'} P_{D'}{}^A, \tag{4.163}$$

$$[s^A, \bar{X}] = t^{D'} P_{D'}{}^A, \tag{4.164}$$

$$[s^A, \bar{Y}] = 0, \tag{4.165}$$

$$[s^A, \bar{Z}] = -r^{D'} P_{D'}{}^A, \tag{4.166}$$

$$[t^A, \bar{X}] = s^{D'} P_{D'}{}^A, \tag{4.167}$$

$$[t^A, \bar{Y}] = -r^{D'} P_{D'}{}^A, \tag{4.168}$$

$$[t^A, \bar{Z}] = 0, \tag{4.169}$$

$$[\lambda^A, \bar{X}] = r^A, \tag{4.170}$$

$$[\lambda^A, \bar{Y}] = -s^A, \tag{4.171}$$

$$[\lambda^A, \bar{Z}] = -t^A, \tag{4.172}$$

$$[X, \bar{X}] = B - \tfrac{2}{3}(\Lambda - M), \tag{4.173}$$

$$[Y, \bar{Y}] = B - \tfrac{2}{3}(\Lambda + 2M + 3), \tag{4.174}$$

$$[Z, \bar{Z}] = B + \tfrac{2}{3}(2\Lambda + M + 3), \tag{4.175}$$

$$[X, \bar{Y}] = 0, \tag{4.176}$$

$$[Y, \bar{Z}] = 0, \tag{4.177}$$

$$[Z, \bar{X}] = 0, \tag{4.178}$$

$$[X, Y] = 0, \tag{4.179}$$

$$[Y, Z] = 0, \tag{4.180}$$

$$[Z, X] = 0, \tag{4.181}$$

$$[B, \Lambda + 1] = 0, \tag{4.182}$$

$$[B, M + 1] = 0, \tag{4.183}$$

$$[\Lambda + 1, M + 1] = 0, \tag{4.184}$$

$$[B, r^A] = -r^A, \tag{4.185}$$

$$[B, s^A] = \tfrac{1}{3}s^A, \tag{4.186}$$

$$[B, t^A] = \tfrac{1}{3}t^A, \tag{4.187}$$

$$[B, \lambda^A = -\tfrac{1}{3}\lambda^A, \tag{4.188}$$

$$[\Lambda + 1, r^A] = 0, \tag{4.189}$$

$$[\Lambda + 1, s^A] = -s^A, \tag{4.190}$$

$$[\Lambda + 1, t^A] = 0, \tag{4.191}$$

$$[\Lambda + 1, \lambda^A] = -\lambda^A, \tag{4.192}$$

$$[M + 1, r^A] = 0, \tag{4.193}$$

$$[M + 1, s^A] = 0, \tag{4.194}$$

$$[M + 1, t^A] = t^A, \tag{4.195}$$

$$[M + 1, \lambda^A] = \lambda^A, \tag{4.196}$$

$$[B, S^a] = 0, \tag{4.197}$$

$$[\Lambda + 1, S^a] = 0, \tag{4.198}$$

$$[M + 1, S^a] = 0, \tag{4.199}$$

$$[B, X] = \tfrac{2}{3}X, \tag{4.200}$$

$$[B, Y] = -\tfrac{2}{3}Y, \tag{4.201}$$

$$[B, Z] = -\tfrac{2}{3}Z, \tag{4.202}$$

$$[\Lambda + 1, X] = -X, \tag{4.203}$$

$$[\Lambda + 1, Y] = 0, \tag{4.204}$$

$$[\Lambda+1, Z] = -Z, \tag{4.205}$$

$$[M+1, X] = X, \tag{4.206}$$

$$[M+1, Y] = Y, \tag{4.207}$$

$$[M+1, Z] = 0. \tag{4.208}$$

COROLLARY B 1. *$\mathcal{Q}$ contains* SO(6) $\oplus$ SO(2), *or equivalently* U(4), *as a compact sub-algebra.*

The proof is by direct construction. Form the four by four Hermitian matrix operator

$$A_i^{j'} \equiv \begin{pmatrix} S^{AA'} - \frac{1}{2}\hbar(B - \frac{1}{3}(M-\Lambda))\,P^{AA'} & \lambda^A & r^A \\ \lambda^{A'} & -\frac{1}{2}\hbar(M-\Lambda) & \hbar\bar{X} \\ r^{A'} & \hbar X & \hbar[2B + \frac{1}{3}(M-\Lambda)] \end{pmatrix}. \tag{4.209}$$

Then with

$$\delta^i_{j'} \equiv \begin{pmatrix} P_{AA'} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.210}$$

and

$$\delta^{j'}_i \equiv \begin{pmatrix} P^{AA'} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.211}$$

one finds, using (4.121)–(4.208), that $A_i^{j'}$ obeys

$$\overline{A_i^{j'}} = A_j^{i'}, \quad A_i^{j'}\,\delta^i_{j'} = 0, \tag{4.212}$$

$$[A_i^{j'}, A_k^{l'}] = -\hbar[\delta_k^{j'}A_i^{l'} - \delta_i^{l'}A_k^{j'}], \tag{4.213}$$

so $A_i^{j'}$ generates the SU(4) Lie algebra which is isomorphic to the SO(6) Lie algebra; $\Lambda + M + 2$ commutes with all of $A_i^{j'}$, so the corollary is proved: the sub-algebra of $\mathcal{Q}$ spanned by S^a, B, $\Lambda+1$, $M+1$, λ^A, r^A, $\lambda^{A'}$, $r^{A'}$, X and $\bar{X}$ is SO(6) $\oplus$ SO(2) $\approx$ U(4). Similarly one may show that $\mathcal{Q}$ contains U(3, 1) and U(2, 2).

A basis for a Cartan sub-algebra of $\mathcal{Q}$ is given by B, $\Lambda+1$, $M+1$, and S_3, a component of S^a: $S^a\alpha_A\bar{\alpha}_{A'}(P^b\alpha_B\bar{\alpha}_{B'})^{-1}$.

Combine B, Λ, M and S_3 into the following quantities:

$$H_0 \equiv -S_3 - \tfrac{1}{2}B - \tfrac{1}{3}(M + 2\Lambda + 3), \tag{4.214}$$

$$H_1 \equiv S_3 - \tfrac{1}{2}B - \tfrac{1}{3}(M + 2\Lambda + 3), \tag{4.215}$$

$$H_2 \equiv -B + \tfrac{1}{3}(M + 2\Lambda), \tag{4.216}$$

$$H_3 \equiv M + 1. \tag{4.217}$$

A root for $\mathcal{Q}$ is an element q of $\mathcal{Q}$ obeying

$$[H_i, q] = \alpha_i q, \tag{4.218}$$

for numbers α_1, α_2, α_3 and α_4.

A root will be called positive if and only if $\alpha_1 > 0$ or $\alpha_1 = 0$, $\alpha_2 > 0$, or $\alpha_1 = \alpha_2 = 0$, $\alpha_3 > 0$, or $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 > 0$. Using a spinor basis, α^A, $\bar{\alpha}^{B'}P_{B'}{}^A$, for the unprimed spinors, one finds that the two components of each of r^A, λ^A, s^A and $t^{A'}$ give positive roots, as do $\bar{X}$, Y, Z and $S_- \equiv S^{A'B}P_B{}^{B'}\bar{\alpha}_{A'}\bar{\alpha}_{B'}$. Then $r^{A'}$, $\lambda^{A'}$, $s^{A'}$, t^A, X, $\bar{Y}$, $\bar{Z}$ and $S_+ \equiv S^{AB'}P_{B'}{}^B\alpha_A\alpha_B$ give negative roots. The various commutators are given in table 1. In table 1, the meaning of, for instance, the entry

$\frac{1}{2}$ at the s^A row, S_3 column, is that S_3 commuted with the first component $s^A\alpha_A$ of s^A gives $\frac{1}{2}s^A\alpha_A$. Similarly for the other entries.

It will be noticed that the α_i root pattern is exactly that of the complex simple Lie algebra, D_4 or SO(8, C). To check that this is a correct identification, one needs to compute the Dynkin diagram of the algebra.

Table 1. System of positive roots of $\mathcal{Q}$

	B	$A+1$	$M+1$	S_3	α_0	α_1	α_2	α_3
r^A	-1	0	0	$\frac{1}{2}$	0	1	1	0
	-1	0	0	$-\frac{1}{2}$	1	0	1	0
λ^A	$-\frac{1}{3}$	-1	1	$\frac{1}{2}$	0	1	0	1
	$-\frac{1}{3}$	-1	1	$-\frac{1}{2}$	1	0	0	1
s^A	$\frac{1}{3}$	-1	0	$\frac{1}{2}$	0	1	-1	0
	$\frac{1}{3}$	-1	0	$-\frac{1}{2}$	1	0	-1	0
$t^{A'}$	$-\frac{1}{3}$	0	-1	$-\frac{1}{2}$	1	0	0	-1
	$-\frac{1}{3}$	0	-1	$\frac{1}{2}$	0	1	0	-1
$\bar{X}$	$-\frac{2}{3}$	1	-1	0	0	0	1	-1
Y	$-\frac{2}{3}$	0	1	0	0	0	1	1
Z	$-\frac{2}{3}$	-1	0	0	1	1	0	0
S_-	0	0	0	-1	1	-1	0	0

With the help of equations (4.121)–(4.208) and table 1, the Killing form is relatively easy to compute, by using the definition

$$\langle q, q' \rangle \equiv Tr(\text{adj}\,(q)\cdot \text{adj}\,(q')) \tag{4.219}$$

giving the scalar product of elements q and q' in $\mathcal{Q}$, adj being the adjoint action of the Lie algebra on itself.

One finds the only non-vanishing scalar products are as follows:

$$\langle B, B\rangle = 8, \quad \langle A, A\rangle = 12, \quad \langle M, M\rangle = 12, \quad \langle A, M\rangle = -6, \tag{4.220}$$

$$\langle r^A, r^{A'}\rangle = \langle s^A, s^{A'}\rangle = \langle t^A, t^{A'}\rangle = \langle \lambda^A, \lambda^{A'}\rangle = 12\hbar^2 P^{AA'}, \tag{4.221}$$

$$\langle S^a, S^b\rangle = 6\hbar^2(P^aP^b - 2g^{ab}), \tag{4.222}$$

$$\langle X, \bar{X}\rangle = -\langle Y, \bar{Y}\rangle = -\langle Z, \bar{Z}\rangle = 12. \tag{4.223}$$

In particular (4.222) gives
$$\langle S_3, S_3\rangle = 6. \tag{4.224}$$

Then (4.220) and (4.224), with (4.214)–(4.217) give

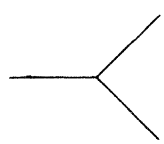
$$\langle H_i, H_j\rangle = 12\delta_{ij}. \tag{4.225}$$

Thus the metric on the Cartan algebra is of the standard Euclidean type. The same is therefore true on the dual space. From table 1, the simple positive roots are given by the following four-vectors:

$$a = (1,\ -1,\ 0,\ 0), \quad b = (0,\ 1,\ -1,\ 0), \tag{4.226}$$

$$c = (0,\ 0,\ 1,\ -1), \quad d = (0,\ 0,\ 1,\ 1), \tag{4.227}$$

The angles ab, bc and bd are each 120°. The angles ac, ad and cd are each 90°. Hence the Dynkin diagram is



(4.228)

and the complexification of $\mathscr{Q}$ is indeed D_4. Hence $\mathscr{Q}$ is a real form of SO(8, C). In particular $\mathscr{Q}$ is simple.

$\mathscr{Q}$ is not SO(4, 4) or SO(5, 3), because it contains SO(6), nor is it SO(7, 1), since it contains SU(3, 1). Hence, by elimination, $\mathscr{Q}$ is SO(6, 2). In particular the U(4) algebra (4.209) is the maximal compact sub-algebra of $\mathscr{Q}$.

One may also see that $\mathscr{Q}$ is SO(6, 2), more constructively, as follows: S^a, λ^A, r^A, $\lambda^{A'}$, $r^{A'}$, X and $\bar{X}$, B, $\Lambda+1$ and $M+1$ generate the algebra $\mathscr{K}$ ($\equiv$ U(4)). Then

$$\mathscr{Q} = \mathscr{K} \oplus \mathscr{P}, \tag{4.229}$$

where $\mathscr{P}$ is generated by s^A, t^A, Y, Z, $s^{A'}$, $t^{A'}$, $\bar{Y}$ and $\bar{Z}$.

From (4.121)–(4.208) one sees that

$$[\mathscr{K}, \mathscr{K}] \subset \mathscr{K}, \quad [\mathscr{K}, \mathscr{P}] \subset \mathscr{P} \quad \text{and} \quad [\mathscr{P}, \mathscr{P}] \subset \mathscr{K}. \tag{4.230}$$

Also from (4.220)–(4.233) the Killing form is positive definite on $\mathscr{K}$, negative definite on $\mathscr{P}$, and $\mathscr{K}$ is orthogonal to $\mathscr{P}$. $\mathscr{K}$ is sixteen dimensional, $\mathscr{P}$ is twelve dimensional. All this information uniquely picks out SO(6, 2). The Weyl unitary trick, replacing $\mathscr{P}$ by i$\mathscr{P}$, would render the algebra compact, hence SO(8).

Summarizing one has:

Corollary B 2. *$\mathscr{Q}$ is the real Lie algebra* SO(6, 2), *of rank four, with a Cartan algebra spanned by B, $\Lambda+1$, $M+1$ and S_3. $\mathscr{Q}$ has Cartan decomposition $\mathscr{K}\oplus\mathscr{P}$, where $\mathscr{K}$ =* U(4), *the maximal compact sub-algebra, is spanned by S^a, B, $\Lambda+1$, $M+1$, X, r^A, λ^A, $\bar{X}$, $r^{A'}$ and $\lambda^{A'}$, and $\mathscr{P}$ is spanned by s^A, t^A, Y, Z, $s^{A'}$, $t^{A'}$, $\bar{Y}$ and $\bar{Z}$. $\mathscr{K}\oplus$i$\mathscr{P}$ is then the associated compact real form* SO(8, R).

The break-up (4.229), satisfying (4.230) deserves further consideration. Equation (4.229) shows that $\mathscr{P}$ is a twelve dimensional $\mathscr{K}$-module. However there is no twelve-dimensional irreducible SU(4) representation, so $\mathscr{P}$ is not irreducible. Starting at (4.121)–(4.208) one sees quickly that $\mathscr{P}$ breaks up into

$$\mathscr{P} = P_+ \oplus P_-, \tag{4.231}$$

where P_+ is spanned by $s^{A'}$, t^A, Y and $\bar{Z}$, P_- by s^A, $t^{A'}$, $\bar{Y}$ and Z, and each of P_+ and P_- form a six-dimensional irreducible representation of SU(4):

$$[\mathscr{K}, P_+] = P_+, \quad [\mathscr{K}, P_-] = P_-. \tag{4.232}$$

Also from (4.121)–(4.208),

$$[P_+, P_+] = 0, \quad [P_+, P_-] = \mathscr{K}, \quad [P_-, P_-] = 0. \tag{4.233}$$

Thinking in terms of SO(6) rather than SU(4), P_+ and P_- transform as the fundamental six-dimensional representation of SO(6), which as an SU(4) representation is represented by the Young tableau ⊟, corresponding to skew tensors of two indices in four-dimensional space.

$\mathscr{K}$ contains the basic SU(3) algebra, S given in (3.177). Each of P_+ and P_- is reducible as an S-module:

$$P_+ = P^+_+ \oplus P^-_+, \tag{4.234}$$

$$P_- = P^+_- \oplus P^-_-, \tag{4.235}$$

where P^+_+ is spanned by t^A and Y, P^-_+ by $s^{A'}$ and $\bar{Z}$, P^-_- by $t^{A'}$ and $\bar{Y}$, and P^+_- by s^A and Z. Each is then either a triplet or conjugate triplet representation of SU(3). One has

$$[S, P^+_+] = P^+_+, \quad [S, P^+_-] = P^+_-, \quad [S, P^-_+] = P^-_+, \quad [S, P^-_-] = P^-_-, \tag{4.236}$$

$$[P^+_+, P^+_+] = [P^-_+, P^+_+] = [P^-_+, P^-_+] = 0, \tag{4.237}$$

$$[P^-_-, P^-_-] = [P^+_-, P^-_-] = [P^+_-, P^+_-] = 0. \tag{4.238}$$

Also under the action of $S, \mathscr{K}$ decomposes as

$$\mathscr{K} = S \oplus N_+ \oplus N_- \oplus C \oplus D, \tag{4.239}$$

where N_+ is the triplet spanned by λ^A and X, N_- the anti-triplet spanned by $\lambda^{A'}$ and $\bar{X}$, and C and D are singlets spanned by $\Lambda - M$ and $\Lambda + M + 2$ respectively.

One has

$$[C, C] = [D, D] = [C, D] = 0, \tag{4.240}$$

$$[C, N_\pm] = N_\pm, \quad [D, N_\pm] = N_\pm, \tag{4.241}$$

$$[N_+, N_+] = [N_-, N_-] = 0, \tag{4.242}$$

$$[N_+, N_-] = S \oplus C, \tag{4.243}$$

$$[S, C] = [S, D] = 0, \quad [S, N_\pm] = N_\pm, \quad [S, S] = S, \tag{4.244}$$

$$[P^+_+, S] = P^+_+, \quad [P^+_+, N_\pm] = N_\pm, \tag{4.245}$$

$$[P^+_+, C] = P^+_+, \quad [P^+_+, D] = P^+_+, \tag{4.246}$$

$$[P^-_+, S] = P^-_+, \quad [P^-_+, N_+] = P^+_+, \quad [P^-_+, N_-] = 0, \tag{4.247}$$

$$[P^-_+, C] = P^-_+, \quad [P^-_+, D] = P^-_+, \tag{4.248}$$

$$[P^-_-, S] = P^-_-, \quad [P^-_-, N_+] = P^+_-, \quad [P^-_-, N_-] = 0, \tag{4.249}$$

$$[P^-_-, C] = P^-_-, \quad [P^-_-, D] = P^-_-, \tag{4.250}$$

$$[P^+_-, S] = P^+_-, \quad [P^+_-, N_+] = 0, \quad [P^+_-, N_-] = P^-_-, \tag{4.251}$$

$$[P^+_-, C] = P^+_-, \quad [P^+_-, D] = P^+_-, \tag{4.252}$$

$$[P^+_+, P^+_-] = N_+, \tag{4.253}$$

$$[P^+_+, P^-_-] = S \oplus (\tfrac{3}{2}D - \tfrac{1}{2}C), \tag{4.254}$$

$$[P^-_+, P^-_-] = N_-, \tag{4.255}$$

$$[P^-_+, P^+_-] = S \oplus (\tfrac{3}{2}D + \tfrac{1}{2}C), \tag{4.256}$$

where by $\frac{3}{2}D - \frac{1}{2}C$ and $\frac{3}{2}D + \frac{1}{2}C$ one means the algebras generated by $(\Lambda + 2M + 3)$ and $(2\Lambda + M + 3)$ respectively.

Notice that the grading given here is a (Λ, M) grading: the algebra $S \oplus C \oplus D$ has weight $(0, 0)$, N_+ has weight $(-1, 1)$, N_- has weight $(1, -1)$, P^+_+ has weight $(0, 1)$, P^-_+ has weight $(1, 0)$, P^-_- has weight $(0, -1)$ and P^+_- has weight $(-1, 0)$.

The triplets (t^A, Y), (s^A, Z) and $(\lambda^{A'}, \bar{X})$ have identical transformation properties under S as do the anti-triplets $(-t^{A'}, \bar{Y})$, $(-s^{A'}, \bar{Z})$ and (λ^A, X).

There are several ways of describing the structure of the algebra $\mathscr{Q}$ more concisely than the presentation given in equations (4.121)–(4.208). Two such descriptions will now be given. The first is tied to the Cartan decomposition. Define

$$L^{k'l'} \equiv \begin{pmatrix} \hbar\epsilon^{KL} Y & -s^{A'} P_{A'}{}^K & t^K \\ s^{A'} P_{A'}{}^L & 0 & -\hbar\bar{Z} \\ -t^L & \hbar\bar{Z} & 0 \end{pmatrix}. \tag{4.257}$$

Then, with $A_i^{j'}$ given by (4.209), one finds, using (4.121)–(4.208),

$$[A_i^{j'}, L^{k'l'}] = -2\hbar\delta_i^{[k'} L^{l']j'} - \tfrac{1}{2}\hbar\delta_i^{j'} L^{k'l'}. \tag{4.258}$$

$L^{k'l'}$ has Hermitian conjugate

$$L_{kl} = \begin{pmatrix} \hbar\epsilon^{K'L'}\bar{Y} & s^A P_A{}^{K'} & -t^{K'} \\ -s^A P_A{}^{L'} & 0 & -\hbar Z \\ t^K & \hbar Z & 0 \end{pmatrix} \quad (4.259)$$

which obeys
$$[A_i^{j'}, L_{kl}] = 2\hbar\delta_{[k}^{j'} L_{l]i} + \tfrac{1}{2}\hbar\delta_i^{j'} L_{kl}. \quad (4.260)$$

Also $L^{k'l'} = -L^{l'k'}$, $L_{kl} = -L_{lk}$ and

$$[L^{k'l'}, L^{m'n'}] = 0, \quad [L_{kl}, L_{mn}] = 0, \quad (4.261)$$

$$[L^{k'l'}, L_{mn}] = -4\hbar\delta_{[m}^{[k'} A_{n]}^{l']} - 2\hbar^2(\Lambda + M + 2)\,\delta_{[m}^{k'}\delta_{n]}^{l'}. \quad (4.262)$$

In view of (4.262), one may define the Hermitian U(4) generators

$$B_i^{j'} \equiv A_i^{j'} + \tfrac{1}{2}\hbar(\Lambda + M + 2)\,\delta_i^{j'}. \quad (4.263)$$

Then $B_i^{j'}$, $L^{k'l'}$ and L_{kl} obey

$$B_i^{j'}\,\delta_{j'}^{i} = 2\hbar(\Lambda + M + 2), \quad \bar{B}_i^{j'} = B_i^{j'}, \quad (4.264)$$

$$[B_i^{j'}, B_k^{l'}] = -\hbar(\delta_k^{j'} B_i^{l'} - \delta_i^{l'} B_k^{j'}), \quad (4.265)$$

$$[\Lambda + M + 2, B_i^{j'}] = 0, \quad (4.266)$$

$$[B_i^{j'}, L^{k'l'}] = -2\hbar\delta_i^{[k'} L^{l']j'}, \quad (4.267)$$

$$[B_i^{j'}, L_{kl}] = 2\hbar\delta_{[k}^{j'} L_{l]i}, \quad (4.268)$$

$$[\Lambda + M + 2, L^{k'l'}] = L^{k'l'}, \quad (4.269)$$

$$[\Lambda + M + 2, L_{kl}] = -L_{kl}, \quad (4.270)$$

$$[L^{k'l'}, L_{kl}] = -4\hbar\delta_{[k}^{[k'} B_{l]}^{l']}. \quad (4.271)$$

The next method exploits the symmetry between r^A, s^A, t^A and $P^A{}_{B'}\lambda^{B'}$. Define

$$x_1{}^A \equiv r^A, \quad x_2{}^A \equiv s^A, \quad x_3{}^A \equiv t^A, \quad x_4{}^A \equiv P^A{}_{B'}\lambda^{B'}, \quad (4.272)$$

$$x_1{}^{A'} \equiv r^{A'}, \quad x_2{}^{A'} \equiv s^{A'}, \quad x_3{}^{A'} \equiv t^{A'}, \quad x_4{}^A \equiv P^{A'}{}_B\lambda^B. \quad (4.273)$$

Then, by using (4.121)–(4.208),

$$[x_i{}^A, x_j{}^{A'}] = 0, \quad \text{if} \quad i \neq j, \quad (4.274)$$

$$[x_i{}^A, x_i{}^{A'}] = \hbar S^a - \tfrac{1}{2}\hbar^2 K_i P^a, \quad (4.275)$$

where the K_i are given by
$$K_1 \equiv 3B, \quad (4.276)$$

$$K_2 \equiv -B + \tfrac{4}{3}(2\Lambda + M + 3), \quad (4.277)$$

$$K_3 \equiv -B - \tfrac{4}{3}(\Lambda + 2M + 3), \quad (4.278)$$

$$K_4 \equiv -B + \tfrac{4}{3}(M - \Lambda). \quad (4.279)$$

Next
$$[K_i, x_j{}^A] = x_j{}^A, \quad [K_i, x_j{}^{A'}] = -x_j{}^{A'}, \quad \text{if} \quad i \neq j, \quad (4.280)$$

$$[K_i, x_i{}^A] = -3x_i{}^A, \quad [K_i, x_i{}^{A'}] = 3x_i{}^{A'}, \quad (4.281)$$

$$[K_i, K_j] = 0. \quad (4.282)$$

Then
$$[x_i{}^A, x_j{}^B] = -\hbar^2\epsilon^{AB}X_{ij}, \quad (4.283)$$

where $X_{ij} = X_{ji}$, $X_{ii} = 0$, for each i:

$$X_{12} = Z, \quad X_{14} = -\bar{X},$$
$$X_{31} = Y, \quad X_{24} = \bar{Y},$$
$$X_{23} = X, \quad X_{34} = \bar{Z}. \tag{4.284}$$

Therefore,
$$[x_i{}^{A'}, x_j{}^{B'}] = -\tfrac{1}{2}\hbar^2 \epsilon_{ijkl} X^{kl} \epsilon^{A'B'}, \tag{4.285}$$

where $\epsilon_{ijkl} = \epsilon_{jikl} = \epsilon_{ijlk}$ and $\epsilon_{ijkl} = 0$, unless i, j, k and l are all unequal, when $\epsilon_{ijkl} = 1$.

Next
$$[X_{12}, X_{34}] = \tfrac{1}{4}(K_1 + K_2 - K_3 - K_4), \tag{4.286}$$
$$[X_{31}, X_{24}] = \tfrac{1}{4}(K_1 + K_3 - K_2 - K_4), \tag{4.287}$$
$$[X_{23}, X_{14}] = \tfrac{1}{4}(K_2 + K_3 - K_1 - K_4), \tag{4.288}$$
$$[X_{ij}, X_{kl}] = 0, \quad \text{unless } i, j, k \text{ and } l \text{ are all unequal}, \tag{4.289}$$
$$[X_{ij}, x_k^B] = \epsilon_{ijkl} P^B_{A'} x_l^{A'}, \tag{4.290}$$
$$[X_{ij}, x_k^{B'}] = 0, \quad \text{if } k \neq i, k \neq j, \tag{4.291}$$
$$[X_{ij}, x_k^{B'}] = P_B^{B'} x_i^B, \quad \text{if } i \neq j, \tag{4.292}$$

Finally,
$$[S^a, X_{ij}] = 0, \quad [S^a, K_i] = 0, \tag{4.293}$$
$$[S^a, x_i{}^B] = \hbar x_i{}^A P^{BA'} - \tfrac{1}{2}\hbar x_i{}^B P^{AA'}, \tag{4.294}$$
$$[S^a, x_i{}^{B'}] = -\hbar x_i{}^{A'} P^{AB'} + \tfrac{1}{2}\hbar x_i{}^{B'} P^{AA'}, \tag{4.295}$$
$$[S^a, S^b] = i\hbar \epsilon^{abcd} P_c S_d, \tag{4.296}$$
$$[K_i, X_{jk}] = 2X_{jk} \quad \text{if} \quad i \neq j, i \neq k, \tag{4.297}$$
$$[K_i, X_{ik}] = -2X_{ik}. \tag{4.298}$$

With this notation, one easily sees the generalization of (4.209): define, for fixed, distinct p and q,

$$A_i^{j'} \equiv \begin{pmatrix} S^{AA'} - \tfrac{1}{8}\hbar(K_p - K_q) P^{AA'} & -P^A_{B'} x_q^{B'} & x_p^A \\ -P^{A'}_B x_q^B & -\tfrac{1}{8}\hbar(3K_q + K_p) & -\hbar X_{pq} \\ x_p^{A'} & \tfrac{1}{2}\hbar\epsilon_{pqfg} X_{fg} & \tfrac{1}{8}\hbar(3K_p + K_q) \end{pmatrix}. \tag{4.299}$$

Then (4.213) holds. In addition, define, for s and t such that no two of p, q, s and t are equal,

$$L^{k'l'} \equiv \begin{pmatrix} \hbar\epsilon^{KL} X_{pt} & -x_s^{A'} P_{A'}{}^K & x_t^K \\ x_s^{A'} P_{A'}{}^L & 0 & -\hbar X_{tq} \\ -x_t^L & \hbar X_{tq} & 0 \end{pmatrix}, \tag{4.300}$$

$$L_{kl} \equiv \begin{pmatrix} \hbar\epsilon^{K'L'} X_{qs} & x_s^A P_A{}^{K'} & -x_t^{K'} \\ -x_s^A P_A{}^{L'} & 0 & -\hbar X_{sp} \\ x_t^{L'} & \hbar X_{ps} & 0 \end{pmatrix}, \tag{4.301}$$

$$B_i^{j'} \equiv A_i^{j'} + \tfrac{1}{8}(K_s - K_t)\,\delta_i^{j'}. \tag{4.302}$$

Then (4.258), (4.260), (4.261), (4.262), (4.267), (4.268) and (4.271) all hold. The analogues of (4.269), (4.270) and (4.266) now read

$$[K_s - K_t, L^{k'l'}] = 4L^{k'l'}, \tag{4.303}$$
$$[K_s - K_t, L_{kl}] = -4L_{kl}, \tag{4.304}$$
$$[K_s - K_t, B_i^{j'}] = 0. \tag{4.305}$$

When $(p, q) = (1, 4)$ or $(4, 1)$, $B_i^{j'}$ generates $U(4)$, when $(p, q) = (2, 3)$ or $(3, 2)$ $B_i^{j'}$ generates $U(2, 2)$. For all other unequal pairs (p, q), $B_i^{j'}$ generates $U(3, 1)$.

Theorem B is a statement about an algebra of operators. To finish this section and summarize the preceding work, here is the corresponding statement about the Hilbert space acted on by the operators.

Theorem C. *The states* $|b, \lambda, \mu, j, q\rangle$ *where* $\lambda \geqslant 0$, $\mu \geqslant 0$, $b - \frac{1}{3}(\mu - \lambda)$, $2j$ *and* $j - \frac{3}{2}b$ *are integers obeying the inequalities* (3.170) *and* (3.171) *with scalar product given by* (3.151) *and* (3.172) *form a Hilbert space*, $\mathscr{H}$. *With the action of* $\mathscr{Q}$ *on* $\mathscr{H}$ *given by equations* (4.35)–(4.40), (4.58)–(4.63), (4.82)–(4.83) *and* (2.24), $\mathscr{H}$ *forms an irreducible Hermitian representation of the simple Lie algebra* $\mathscr{Q}$ (SO(6, 2)). $\mathscr{H}$ *breaks up into the direct sum of representations* $\mathscr{H}_k$ *of* SO(6) *corresponding to symmetric trace-free tensors of k indices in six dimensions, one for each non-negative integer k; k is then equal to* $\lambda + \mu$.

For theorem C, one must check that the action of $\mathscr{Q}$ behaves correctly at the boundaries of the representation. This is easy to do. However another representation of the Hilbert space $\mathscr{H}$ is available, which renders this checking almost trivial. Since this representation is of some physical interest, it will now be given. It involves a transition from the (b, λ, μ) basis, to a 'quark', 'di-quark' basis. States will be labelled by q, $\tilde{q}$, d, $\tilde{d}$ and t where q, $\tilde{q}$, d and $\tilde{d}$ are, respectively, the occupation number of 'quarks', 'anti-quarks', 'diquarks' and 'anti-diquarks'. The labels q, $\tilde{q}$, d, $\tilde{d}$ and t are related to b, λ, μ, j and q by the following translation table:

$$b \equiv \tfrac{1}{3}(q - \tilde{q}) + \tfrac{2}{3}(d - \tilde{d}), \tag{4.306}$$

$$\lambda \equiv \tilde{q} + d, \tag{4.307}$$

$$\mu \equiv q + \tilde{d}, \tag{4.308}$$

$$2j \equiv q + \tilde{q}, \tag{4.309}$$

$$q \equiv t; \tag{4.310}$$

with inverse translation

$$q \equiv \tfrac{1}{2}b + j + \tfrac{1}{3}(\mu - \lambda), \tag{4.311}$$

$$\tilde{q} \equiv -\tfrac{1}{2}b + j - \tfrac{1}{3}(\mu - \lambda), \tag{4.312}$$

$$d \equiv \tfrac{1}{2}b - j + \tfrac{1}{3}(2\lambda + \mu), \tag{4.313}$$

$$\tilde{d} \equiv -\tfrac{1}{2}b - j + \tfrac{1}{3}(\lambda + 2\mu), \tag{4.314}$$

$$t \equiv q. \tag{4.315}$$

From (3.171) and (4.311)–(4.314), one sees that q, $\tilde{q}$, d and $\tilde{d}$ are non-negative integers. Thus the boundaries of the representation are determined by the vanishing of one or more of the quantities q, $\tilde{q}$, d and $\tilde{d}$.

Looking at (4.306)–(4.309), one sees that the quantum numbers of the q, $\tilde{q}$, d, $\tilde{d}$ system are just those of a system of q quarks, $\tilde{q}$ anti-quarks, d di-quarks and $\tilde{d}$ anti-diquarks, where the quark is thought of as an SU(3) triplet contributing 1 to μ and 0 to λ, of spin $\frac{1}{2}$ and baryon number $\frac{1}{3}$. The anti-quark contributes as an anti-triplet: 0 to μ, 1 to λ, of spin $\frac{1}{2}$ and baryon number $-\frac{1}{3}$. The diquark contributes as a 'bound state' of two quarks, where the SU(3) representations have combined to form an anti-triplet, the spins have interfered destructively to give spin zero and the baryon numbers have added, to give $\frac{2}{3}$. Similarly for the anti-diquark. Given the quantum numbers for the quark, anti-quark, diquark and anti-diquark the quantum numbers for the whole system just add constructively.

The $\mathscr{Q}$ representation now reads

$$\langle q, \tilde{q}, d, \tilde{d}, t | q, \tilde{q}, d, \tilde{d}, t\rangle = (\bar{\alpha}^D P_{DD'}\,\alpha^{D'})^{q+\tilde{q}}\, t!\,[(q+\tilde{q}+1)\,(q+\tilde{q}-t)!]^{-1} \times d!\,\tilde{d}!\,q!\,\tilde{q}!\,(d+q+\tilde{q}+1)!\,(\tilde{d}+q+\tilde{q}+1)!, \tag{4.316}$$

$$\hbar^{-1}(\bar{\alpha}^D P_{DD'}\alpha^{D'})\,S^a|q, \tilde{q}, d, \tilde{d}, t\rangle = \alpha^{A'}\alpha^{B'}P_{B'}{}^A|q, \tilde{q}, d, \tilde{d}, t+1\rangle + \tfrac{1}{2}(q+\tilde{q}-t)\,(\bar{\alpha}^A\alpha^{A'} - \alpha^{B'}P_{B'}{}^A\,\bar{\alpha}^B P_B{}^{A'})|q, \tilde{q}, d, \tilde{d}, t\rangle + t(q+\tilde{q}-t+1)\,\bar{\alpha}^A\bar{\alpha}^B P_B{}^{A'}|q, \tilde{q}, d, \tilde{d}, t-1\rangle, \tag{4.317}$$

$$l^A|q, \tilde{q}, d, \tilde{d}, t\rangle = (\bar{\alpha}^D P_{DD'}\alpha^{D'})^{-1}[\alpha^{B'}P_{B'}{}^A|q, \tilde{q}+1, d-1, \tilde{d}, t+1\rangle + (q+\tilde{q}+1-t)\,\bar{\alpha}^A|q, \tilde{q}+1, d-1, \tilde{d}, t\rangle], \tag{4.318}$$

$$m^A|q, \tilde{q}, d, \tilde{d}, t\rangle = P_{B'}{}^A\alpha^{B'}|q-1, \tilde{q}, d, \tilde{d}+1, t\rangle - t\bar{\alpha}^A|q-1, \tilde{q}, d, \tilde{d}+1, t-1\rangle, \tag{4.319}$$

$$l^{A'}|q, \tilde{q}, d, \tilde{d}, t\rangle = \alpha^{A'}|q, \tilde{q}-1, d+1, \tilde{d}, t\rangle + tP_B{}^{A'}\bar{\alpha}^B|q, \tilde{q}-1, d+1, \tilde{d}, t-1\rangle, \tag{4.320}$$

$$m^{A'}|q, \tilde{q}, d, \tilde{d}, t\rangle = (\bar{\alpha}^D P_{DD'}\alpha^{D'})^{-1}[-\alpha^{A'}|q+1, \tilde{q}, d, \tilde{d}-1, t-1\rangle + (q+\tilde{q}+1-t)\,P_B{}^{A'}\bar{\alpha}^B|q+1, \tilde{q}, d, \tilde{d}-1, t\rangle], \tag{4.321}$$

$$\rho|q, \tilde{q}, d, \tilde{d}, t\rangle = |q+1, \tilde{q}-1, d, \tilde{d}-1, t\rangle, \tag{4.322}$$

$$\sigma|q, \tilde{q}, d, \tilde{d}, t\rangle = |q+1, \tilde{q}-1, d+1, \tilde{d}, t\rangle, \tag{4.323}$$

$$\tau|q, \tilde{q}, d, \tilde{d}, t\rangle = |q+1, \tilde{q}-1, d, \tilde{d}, t\rangle, \tag{4.324}$$

$$\bar{\rho} = \rho^{-1}Q[(\tilde{D}+Q+\tilde{Q}+2)\,(\tilde{Q}+1)\,(\tilde{D}+1)]^{-1}, \tag{4.325}$$

$$\bar{\sigma} = \sigma^{-1}QD(D+Q+\tilde{Q}+1)\,(\tilde{Q}+1)^{-1}, \tag{4.326}$$

$$\bar{\tau} = \tau^{-1}Q(\tilde{Q}+1)^{-1}, \tag{4.327}$$

$$X = \tau\tilde{Q}, \tag{4.328}$$

$$Y = \rho^{-1}\tau, \tag{4.329}$$

$$Z = -\sigma^{-1}\tau D(D+Q+\tilde{Q}+1), \tag{4.330}$$

$$\bar{X} = \tau^{-1}Q, \tag{4.331}$$

$$\bar{Y} = \rho\tau^{-1}\tilde{D}(\tilde{D}+Q+\tilde{Q}+1), \tag{4.332}$$

$$\bar{Z} = -\sigma\tau^{-1}, \tag{4.333}$$

$$\bar{l}^{A'} = l^{A'}(Q+\tilde{Q})\,(D+Q+\tilde{Q}+1)\,\tilde{Q}[(Q+\tilde{Q}+1)\,(D+1)]^{-1}, \tag{4.334}$$

$$\bar{m}^{A'} = m^{A'}(Q+\tilde{Q}+2)\,\tilde{D}[(Q+\tilde{Q}+1)\,(D+Q+\tilde{Q}+2)\,(Q+1)]^{-1}, \tag{4.335}$$

$$r^A = \hbar l^A D(Q+\tilde{Q}+1)^{-1} + \hbar m^A Q(D+Q+\tilde{Q}+1)\,(Q+\tilde{Q}+1)^{-1}, \tag{4.336}$$

$$s^A = -\hbar\rho l^A D\tilde{D}(Q+\tilde{Q}+1)^{-1} + \hbar\rho m^A\tilde{Q}(D+Q+\tilde{Q}+1) \times (\tilde{D}+Q+\tilde{Q}+1)\,(Q+\tilde{Q}+1)^{-1}, \tag{4.337}$$

$$t^A = -\hbar\sigma l^A(Q+\tilde{Q}+1)^{-1} + \hbar\sigma m^A\tilde{Q}(Q+\tilde{Q}+1)^{-1}, \tag{4.338}$$

$$\lambda^A = -\hbar\tau l^A D(Q+\tilde{Q}+1)^{-1} + \hbar\tau m^A\tilde{Q}(D+Q+\tilde{Q}+1)\,(Q+\tilde{Q}+1)^{-1}, \tag{4.339}$$

$$r^{A'} = \hbar l^{A'}\tilde{Q}(\tilde{D}+Q+\tilde{Q}+1)\,(Q+\tilde{Q}+1)^{-1} + \hbar m^{A'}\tilde{D}(Q+\tilde{Q}+1)^{-1}, \tag{4.340}$$

$$s^{A'} = \hbar\rho^{-1}l^{A'}Q(Q+\tilde{Q}+1)^{-1} - \hbar\rho^{-1}m^{A'}(Q+\tilde{Q}+1)^{-1}, \tag{4.341}$$

$$t^{A'} = \hbar\sigma^{-1} l^{A'}(\tilde{D}+Q+\tilde{Q}+1)\,(D+Q+\tilde{Q}+1)\,Q(Q+\tilde{Q}+1)^{-1} - \hbar\sigma^{-1} m^{A'} D\tilde{D}(Q+\tilde{Q}+1)^{-1}, \qquad (4.342)$$

$$\lambda^{A'} = -\hbar\tau^{-1} l^{A'} Q(\tilde{D}+Q+\tilde{Q}+1)\,(Q+\tilde{Q}+1)^{-1} + \hbar\tau^{-1} m^{A'} \tilde{D}(Q+\tilde{Q}+1)^{-1}. \qquad (4.343)$$

The effects of the various operators on q, $\tilde{q}$, d and $\tilde{d}$ are summarized in table 2.

Table 2. Shifts in 'quark' content caused by the various operators of the algebra $\mathscr{Q}$

		Δq	$\Delta \tilde{q}$	Δd	$\Delta \tilde{d}$
	X	1	−1	0	0
	Y	0	0	0	1
	Z	0	0	−1	0
l^A terms	r^A	0	1	−1	0
	s^A	1	0	−1	−1
	t^A	1	0	0	0
	λ^A	1	0	−1	0
m^A terms	r^A	−1	0	0	1
	s^A	0	−1	0	0
	t^A	0	−1	1	1
	λ^A	0	−1	0	1
	$\bar{X}$	−1	1	0	0
	$\bar{Y}$	0	0	0	−1
	$\bar{Z}$	0	0	1	0
$l^{A'}$ terms	$r^{A'}$	0	−1	1	0
	$s^{A'}$	−1	0	1	1
	$t^{A'}$	−1	0	0	0
	$\lambda^{A'}$	−1	0	1	0
$m^{A'}$ terms	$r^{A'}$	1	0	0	−1
	$s^{A'}$	0	1	0	0
	$t^{A'}$	0	1	−1	−1
	$\lambda^{A'}$	0	1	0	−1

A quick inspection shows that whenever −1 appears in table 2, the coefficient of the appropriate piece vanishes at the boundary of the representation, as required.

The 'quark'–'diquark' representation of the system perhaps reveals most clearly the heart of the present theory. If one thinks of the quarks and diquarks as real physical entities, then there are two quite distinct mechanisms for raising the spin of the system – one by adding relative orbital angular momentum to the quarks and diquarks, the other by adding more quarks. Here the first option is simply excluded.

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